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CONVERSE THEOREM ON A GLOBAL CONTRACTION METRIC FOR A PERIODIC ORBIT

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Abstract. Contraction analysis uses a local criterion to prove the long-term behaviour of a dynamical system. A contraction metric is a Riemannian metric with respect to which the distance between adjacent solutions contracts. If adjacent solutions in all directions perpendicular to the flow are contracted, then there exists a unique periodic orbit, which is exponentially stable and we obtain an upper bound on the rate of exponential attraction.

In this paper we study the converse question and show that, given an exponentially stable periodic orbit, a contraction metric exists on its basin of attraction and we can recover the upper bound on the rate of exponential attraction.

1. **Introduction.** The stability and the basin of attraction of periodic orbits provide important information in many applications. However, already the determination of a periodic orbit is a non-trivial task as it involves solving the differential equation. There are several methods to study the stability and the basin of attraction of periodic orbits, among them Lyapunov functions and contraction metrics. A Lyapunov function, as well as the classical definition of stability, requires the knowledge of the position of the periodic orbit which in many applications can only be approximated. Contraction analysis, on the other hand, does not require us to know the location of the periodic orbit.

Throughout the paper we will study the autonomous ODE

$$\dot{\mathbf{x}} \quad = \quad \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ with $\sigma \geq 1$. We denote the solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ by $S_t\mathbf{x}_0 = \mathbf{x}(t)$ and assume that it exist for all $t \geq 0$.

We treat $\mathbb{R}^n$ as a Riemannian manifold, equipped with a Riemannian metric, which can be expressed by a matrix-valued function $M(\mathbf{x})$. In particular, $M(\mathbf{x})$ defines a point-dependent scalar product through $\langle \mathbf{v}, \mathbf{w}\rangle_{M(\mathbf{x})} = \mathbf{v}^T M(\mathbf{x})\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ from the tangent space at $\mathbf{x}$. The rate of expansion over time of the distance between solutions of (1) through $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{v}$ for small $\delta > 0$ with respect to the Riemannian metric is expressed by $L_M(\mathbf{x}; \mathbf{v})$, see (2). The distance between the solution through $\mathbf{x}$ and adjacent solutions through $\mathbf{x} + \mathbf{v}$ is decreasing if $L_M(\mathbf{x}; \mathbf{v}) < 0$. If the distance between $\mathbf{x}$ and all adjacent solutions in direction

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$\mathbf{v}$ perpendicular to the direction of the flow is decreasing, expressed by $L_M(\mathbf{x}) < 0$, see (3), then M is called a contraction metric. Note that for a periodic orbit, contraction cannot occur in direction of the flow.

Definition 1.1 (Contraction metric)**.** A Riemannian metric is a function $M \in C^0(G, \mathbb{S}^n)$, where $G \subset \mathbb{R}^n$ is open and $\mathbb{S}^n$ denotes the symmetric $n \times n$ matrices, such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in G$ and the orbital derivative of M,

$$M'(\mathbf{x}) = \frac{d}{dt} M(S_t\mathbf{x})\big|_{t=0},$$

exists for all $\mathbf{x} \in G$ and is continuous.

A sufficient condition for the latter is that $M \in C^1(G, \mathbb{S}^n)$; then $M'_{ij}(\mathbf{x}) = \nabla M_{ij}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ for all $i, j \in \{1, \ldots, n\}$.

Define

$$L_M(\mathbf{x};\mathbf{v}) \quad := \quad \frac{1}{2}\mathbf{v}^T \left(M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})\right)\mathbf{v}. \tag{2}$$

The Riemannian metric M is called **contraction metric in** $K \subset G$ **with exponent** $-\nu < 0$ if $L_M(\mathbf{x}) \le -\nu$ for all $\mathbf{x} \in K$, where

$$L_M(\mathbf{x}) \quad := \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})=0} L_M(\mathbf{x};\mathbf{v}). \tag{3}$$

Note that $L_M(\cdot)$ is a continuous function and, as we will show in the paper, also locally Lipschitz-continuous. Due to the maximum, however, it is not differentiable in general.

The following theorem shows that the existence of a contraction metric implies the existence, uniqueness and stability of a periodic orbit. Moreover, it provides information about its basin of attraction. Note that the conditions on M are local and can easily be checked for a given function M, while the implications are global.

Theorem 1.2. *Let $\varnothing \neq K \subset \mathbb{R}^n$ be a compact, connected and positively invariant set which contains no equilibrium. Let M be a contraction metric in K with exponent $-\nu < 0$, see Definition 1.1.*

Then there exists one and only one periodic orbit $\Omega \subset K$. This periodic orbit is exponentially stable, and the real parts of all Floquet exponents – except the trivial one – are less than or equal to $-\nu$. Moreover, the basin of attraction $A(\Omega)$ contains K.

This theorem goes back to Borg [2] with $M(\mathbf{x}) = I$, and has been extended to a general Riemannian metric [13]. For more results on contraction analysis for a periodic orbit see [9, 8, 10, 11].

Note that a similar result holds with an equilibrium if the contraction takes place in all directions $\mathbf{v}$, i.e. if $L_M(\mathbf{x}) \le -\nu$ in (3) is replaced by $\mathcal{L}_M(\mathbf{x}) := \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}) \le -\nu$. For more references on contraction analysis see [12], and for the relation to Finsler-Lyapunov functions see [4]. The theory of normally hyperbolic invariant manifolds [14] considers more general invariant manifolds, not necessarily attracting, and studies their persistence under perturbations of the underlying system.

In this paper we are interested in global converse results, i.e. given an exponentially stable periodic orbit, does a Riemannian contraction metric as in Definition 1.1 exist in the whole basin of attraction? [12] gives a converse theorem, but here $M(t, \mathbf{x})$ depends on t and will, in general, become unbounded as $t \to \infty$. In [6] the existence of such a contraction metric was shown on a given compact subset of

$A(\Omega)$, first on the periodic orbit, using Floquet theory, and then on K, using a Lyapunov function. The local construction, however, contained an error: the Floquet representation of solutions of the first variational equation along the periodic orbit is in general complex. We will show in this paper, that, by choosing the complex Floquet representation appropriately, the constructed Riemannian metric is real-valued. Moreover, we will show the existence of a Riemannian metric on the whole, possibly unbounded basin of attraction by using a new construction. The Riemannian metric will be arbitrarily close to the true rate of exponential attraction. Let us summarize the main result of the paper in the following theorem.

Theorem 1.3. *Let Ω be an exponentially stable periodic orbit with basin of attraction $A(\Omega)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, let $-\nu < 0$ be the largest real part of all its non-trivial Floquet exponents and $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ with $\sigma \geq 3$.*

Then for all $\epsilon \in (0, \nu/2)$ there exists a contraction metric $M\colon A(\Omega) \to \mathbb{S}^n$ in $A(\Omega)$ as in Definition 1.1 with exponent $-\nu + \epsilon < 0$, i.e.

$$\begin{aligned} L_M(\mathbf{x}) &= \frac{1}{2} \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})=0} \mathbf{v}^T \left(M(\mathbf{x}) D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x}) \right) \mathbf{v} \\ &\leq -\nu + \epsilon \qquad (4) \end{aligned}$$

holds for all $\mathbf{x} \in A(\Omega)$.

The metric is constructed in several steps: first on the periodic orbit, then in a neighborhood, and finally in the whole basin of attraction. In the proof, we define a projection of points $\mathbf{x}$ in a neighborhood of the periodic orbit onto the periodic orbit, namely onto $\mathbf{p} \in \Omega$, such that to $(\mathbf{x} - \mathbf{p})^T M(\mathbf{p}) \mathbf{f}(\mathbf{p}) = 0$. This is then used to synchronize the times of solutions through $\mathbf{x}$ and $\mathbf{p}$, and to define a time-dependent distance between these solutions, which decreases exponentially.

Let us compare our result with other converse theorems for a contraction metric: In [6], a contraction metric on a given compact subset of the basin of attraction was constructed, while in this paper we construct one on the whole basin of attraction. In [5], a construction metric is characterized as the solution of a linear matrix-valued PDE. This is beneficial for its computation by solving the PDE, however, the exponential rate of attraction cannot be recovered, which is an advantage of the approach in this paper. Similar converse theorems for a contraction metric for an equilibrium were obtained in [7] in Theorems 4.1, 4.2 and 4.4.

Let us give an overview over the paper: In Section 2 we prove a special Floquet normal form to ensure that the contraction metric that we later construct on the periodic orbit is real-valued. In Section 3 we prove the main result of the paper, Theorem 1.3, showing the existence of a Riemannian metric on the whole basin of attraction. In the appendix we prove that L_M is locally Lipschitz-continuous.

2. Floquet normal form.

Before we consider the Floquet normal form, we will prove a lemma which calculates $L_M(\mathbf{x})$ for the Riemannian metric $M(\mathbf{x}) = e^{2V(\mathbf{x})} N(\mathbf{x})$.

Lemma 2.1. *Let $N\colon \mathbb{R}^n \to \mathbb{S}^n$ be a Riemannian metric and $V\colon \mathbb{R}^n \to \mathbb{R}$ a continuous function such that the orbital derivative V' exists and is continuous.*

Then $M(\mathbf{x}) = e^{2V(\mathbf{x})} N(\mathbf{x})$ is a Riemannian metric and

$$L_M(\mathbf{x}) = L_N(\mathbf{x}) + V'(\mathbf{x}).$$

Proof. It is clear that $M(\mathbf{x})$ is a positive definite for all $\mathbf{x}$ since $e^{2V(\mathbf{x})} > 0$. We have

$$\begin{aligned} L_M(\mathbf{x};\mathbf{v}) &= \frac{1}{2}\mathbf{v}^T\left(M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x})\right)\mathbf{v} \\ &= \frac{1}{2}\mathbf{v}^T\Bigg(e^{2V(\mathbf{x})}N(\mathbf{x})D\mathbf{f}(\mathbf{x}) + e^{2V(\mathbf{x})}D\mathbf{f}(\mathbf{x})^T N(\mathbf{x}) \\ &\qquad + e^{2V(\mathbf{x})}(2V'(\mathbf{x})N(\mathbf{x}) + N'(\mathbf{x}))\Bigg)\mathbf{v} \\ &= \frac{1}{2}\mathbf{w}^T\left(N(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T N(\mathbf{x}) + N'(\mathbf{x})\right)\mathbf{w} + \mathbf{w}^T N(\mathbf{x})\mathbf{w}\, V'(\mathbf{x}) \end{aligned}$$

with $\mathbf{w} = e^{V(\mathbf{x})}\mathbf{v}$, so $L_M(\mathbf{x};\mathbf{v}) = L_N(\mathbf{x};\mathbf{w}) + \mathbf{w}^T N(\mathbf{x})\mathbf{w}\, V'(\mathbf{x})$. Thus,

$$\begin{aligned} L_M(\mathbf{x}) &= \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x})=0} L_M(\mathbf{x};\mathbf{v}) \\ &= \max_{\mathbf{w}^T N(\mathbf{x})\mathbf{w}=1, \mathbf{w}^T N(\mathbf{x})\mathbf{f}(\mathbf{x})=0} \left[L_N(\mathbf{x};\mathbf{w}) + \mathbf{w}^T N(\mathbf{x})\mathbf{w}V'(\mathbf{x})\right] \\ &= L_N(\mathbf{x}) + V'(\mathbf{x}). \end{aligned}$$

This shows the lemma. □

In order to show later that our constructed Riemannian metric M is real-valued, we will construct a special Floquet normal form in Proposition 1 such that the matrix in (6) is real-valued. In Corollary 1 we will show estimates in the case that (5) is the first variational equation of a periodic orbit. The proof of the following proposition is inspired by [3]. In the following, we denote $A^* = \overline{A^T}$ for a matrix $A \in \mathbb{C}^{n\times n}$.

Proposition 1. *Consider the periodic differential equation*

$$\dot{\mathbf{y}} = F(t)\mathbf{y} \tag{5}$$

where $F \in C^s(\mathbb{R}, \mathbb{R}^{n\times n})$ is T-periodic, $s \geq 1$ and denote by $\Phi \in C^s(\mathbb{R}, \mathbb{R}^{n\times n})$ its principal fundamental matrix solution with $\Phi(0) = I$.

Then there exists a T-periodic function $P \in C^s(\mathbb{R}, \mathbb{C}^{n\times n})$ with $P(0) = P(T) = I$ and a matrix $B \in \mathbb{C}^{n\times n}$ such that for all $t \in \mathbb{R}$

$$\Phi(t) = P(t)e^{Bt}.$$

Denote by $\lambda_1, \ldots, \lambda_r \in \mathbb{R}\setminus\{0\}$ the pairwise distinct real eigenvalues and by $\lambda_{r+1}, \overline{\lambda_{r+1}}, \ldots, \lambda_{r+c}, \overline{\lambda_{r+c}} \in \mathbb{C}\setminus\mathbb{R}$ the pairwise distinct pairs of complex conjugate complex eigenvalues of $\Phi(T)$ with algebraic multiplicity m_j of λ_j. For $\epsilon > 0$ there exists a non-singular matrix $S \in \mathbb{R}^{n\times n}$ such that $B = SAS^{-1}$ with $A =$ $\mathrm{blockdiag}(K_1, K_2, \ldots, K_{r+c})$ and $K_j \in \mathbb{C}^{m_j\times m_j}$ for $j = 1, \ldots, r$ and $K_j \in \mathbb{R}^{2m_j\times 2m_j}$ for $j = r+1, \ldots, r+c$ as well as

$$\frac{1}{2}\mathbf{w}^*(A^* + A)\mathbf{w} \leq \sum_{j=1}^{r+c} c_j \sum_{i=1}^{m_j} \left|w_{i+\sum_{k=1}^{j-1} m_k}\right|^2 \text{ for all } \mathbf{w} \in \mathbb{C}^n,$$

where $c_j = \left(\frac{\ln|\lambda_j|}{T} + \epsilon\right)$ if $m_j \geq 2$ and $c_j = \frac{\ln|\lambda_j|}{T}$ if $m_j = 1$.

Moreover, we have

$$(P^{-1}(t))^*(S^{-1})^* S^{-1} P^{-1}(t) \in \mathbb{R}^{n\times n} \tag{6}$$

for all $t \in \mathbb{R}$.

Proof. Since $F \in C^s(\mathbb{R}, \mathbb{R}^{n\times n})$, we also have $\Phi \in C^s(\mathbb{R}, \mathbb{R}^{n\times n})$. Noting that $\Psi(t) := \Phi(t+T)$ solves (5) with $\Psi(0) = \Phi(T)$, we obtain from the uniqueness of solutions that

$$\Phi(t+T) \quad = \quad \Psi(t) = \Phi(t)\Phi(T) \text{ for all } t \in \mathbb{R}. \tag{7}$$

Consider $C := \Phi(T) \in \mathbb{R}^{n\times n}$ which is non-singular and hence all eigenvalues of $\Phi(T)$ are non-zero. Let $\epsilon' := \frac{1}{2}\min\left(\frac{\epsilon T}{2}, 1\right)$ and $S \in \mathbb{R}^{n\times n}$ be such that $S^{-1}CS =: J$ is in real Jordan normal form with the 1 replaced by $\epsilon'|\lambda_j|$ for each eigenvalue λ_j, i.e. J is a block-diagonal matrix with blocks J_j of the form

$$J_j = \begin{pmatrix} \lambda_j & \epsilon'|\lambda_j| & & & \\ & \lambda_j & \epsilon'|\lambda_j| & & \\ & & \ddots & \ddots & \\ & & & \lambda_j & \epsilon'|\lambda_j| \\ & & & & \lambda_j \end{pmatrix} \in \mathbb{R}^{m_j\times m_j} \text{ for real eigenvalues } \lambda_j \text{ of } C$$

$$\text{and } J_j = \begin{pmatrix} \alpha_j & -\beta_j & \epsilon' r_j & & & & & \\ \beta_j & \alpha_j & & \epsilon' r_j & & & & \\ & & \ddots & & \ddots & & & \\ & & & \alpha_j & -\beta_j & \epsilon' r_j & & \\ & & & \beta_j & \alpha_j & & \epsilon' r_j \\ & & & & & \alpha_j & -\beta_j \\ & & & & & \beta_j & \alpha_j \end{pmatrix} \in \mathbb{R}^{2m_j\times 2m_j} \text{ for each pair}$$

of complex eigenvalues $\alpha_j \pm i\beta_j$ of C, where $r_j = \sqrt{\alpha_j^2 + \beta_j^2}$ and m_j denotes the dimension of the generalized eigenspace of one of them; note that we have pairs of complex conjugate eigenvalues since C is real.

This can be achieved by letting $S_1 \in \mathbb{R}^{n\times n}$ be an invertible matrix such that $S_1^{-1}CS_1$ is the standard real Jordan Normal Form with 1 on the super diagonal. Then define S_2 to be a matrix of blocks

$$\mathrm{diag}(1, \epsilon'|\lambda_j|, (\epsilon')^2|\lambda_j|^2, \ldots, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1})$$

for real λ_j and

$$\mathrm{diag}(1, 1, \epsilon'|\lambda_j|, \epsilon'|\lambda_j|, \ldots, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1}, (\epsilon')^{m_j-1}|\lambda_j|^{m_j-1})$$

for a pair of complex conjugate eigenvalues λ_j and $\overline{\lambda_j}$. Setting $S = S_1S_2$ yields the result.

For each of the blocks, we will now construct a matrix $K_j \in \mathbb{C}^{m_j\times m_j}$ for real eigenvalues λ_j and $K_j \in \mathbb{R}^{2m_j\times 2m_j}$ for each pair of complex eigenvalues $\alpha_j \pm i\beta_j$ such that

$$e^{K_jT} = J_j,$$

which shows with $B = SAS^{-1}$, where $A := \mathrm{blockdiag}(K_1, \ldots, K_r)$, that

$$\begin{aligned} e^{BT} &= Se^{AT}S^{-1} = S\,\mathrm{blockdiag}(e^{K_1T}, \ldots, e^{K_rT})S^{-1} \\ &= SJS^{-1} = C = \Phi(T). \end{aligned} \tag{8}$$

We distinguish between three cases: λ_j being real positive, real negative or complex. Using the series expansion of $\ln(1+x)$ we obtain for a nilpotent matrix $M \in \mathbb{R}^{n\times n}$

$$\exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} M^k\right) \quad = \quad I + M; \tag{9}$$

note that the sum is actually finite.

Case 1: $\lambda_j \in \mathbb{R}^+$

Writing $J_j = \lambda_j(I + \epsilon' N)$ with the nilpotent matrix $N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{m_j \times m_j}$, we define

$$K_j = \frac{1}{T}\left((\ln \lambda_j)I + \sum_{k=1}^{m_j-1} \frac{(-1)^{k+1}}{k}(\epsilon')^k N^k\right) \in \mathbb{R}^{m_j \times m_j}.$$

Since I and N commute, we have with (9) and $N^k = 0$ for $k \geq m_j$

$$\exp(K_j T) = \lambda_j\,(I + \epsilon' N) = J_j.$$

Case 2: $\lambda_j \in \mathbb{R}^-$

With the nilpotent matrix $N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{m_j \times m_j}$ we write $J_j = -|\lambda_j|(I - \epsilon' N)$ and define

$$K_j = \frac{1}{T}\left((i\pi + \ln|\lambda_j|)I + \sum_{k=1}^{m_j-1} \frac{(-1)^{k+1}}{k}(-\epsilon')^k N^k\right) \in \mathbb{C}^{m_j \times m_j}.$$

Since I and N commute, and $N^k = 0$ for $k \geq m_j$ we have with (9)

$$\exp(K_j T) = -|\lambda_j|\,(I - \epsilon' N) = J_j.$$

Case 3: $\lambda_j = \alpha_j + i\beta_j$ **with** $\beta_j \neq 0$

We only consider one of the two complex conjugate eigenvalues λ_j and $\overline{\lambda_j}$ of $\Phi(T)$. Writing λ_j in polar coordinates gives $\lambda_j = \alpha_j + i\beta_j = r_j e^{i\theta_j} = r_j \cos\theta_j + i r_j \sin\theta_j$ with $r_j > 0$ and $\theta_j \in (0, 2\pi)$. Then, defining $R_j = r_j \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}$, $\mathcal{R} = \operatorname{blockdiag}(R_j, R_j, \ldots, R_j) \in \mathbb{R}^{2m_j \times 2m_j}$ and the nilpotent matrix $\mathcal{N} \in \mathbb{R}^{2m_j \times 2m_j}$ having 2×2 blocks of $\begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}^{-1}$ above its diagonal, we have $J_j = \mathcal{R}(I + \epsilon' \mathcal{N})$. We define $\Theta = \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}$ and

$$K_j = \frac{1}{T}\left((\ln r_j)I + \operatorname{blockdiag}(\Theta, \Theta, \ldots, \Theta) + \sum_{k=1}^{2m_j-1} \frac{(-1)^{k+1}}{k}(\epsilon')^k \mathcal{N}^k\right) \in \mathbb{R}^{2m_j \times 2m_j}.$$

Since I, blockdiag$(\Theta,\Theta,\ldots,\Theta)$ and $\mathcal{N}$ commute, we have, using $\mathcal{N}^k=0$ for $k\geq 2m_j$ and (9)

$$\begin{aligned}&\exp(K_jT)\\ &= r_j\,\text{blockdiag}\left(\begin{pmatrix}\cos\theta_j & -\sin\theta_j\\ \sin\theta_j & \cos\theta_j\end{pmatrix},\ldots,\begin{pmatrix}\cos\theta_j & -\sin\theta_j\\ \sin\theta_j & \cos\theta_j\end{pmatrix}\right)(I+\epsilon'\mathcal{N})\\ &= J_j.\end{aligned}$$

We can now define $P\in C^s(\mathbb{R},\mathbb{C}^{n\times n})$ by $P(t)=\Phi(t)e^{-Bt}$, which satisfies $P(0)=I$ and

$$\begin{aligned}P(t+T) &= \Phi(t+T)e^{-BT}e^{-Bt}\\ &= \Phi(t)\Phi(T)e^{-BT}e^{-Bt} \text{ by } (7)\\ &= P(t) \text{ by } (8)\end{aligned}$$

for all $t\geq 0$, so in particular $P(T)=P(0)=I$. We can now write

$$\Phi(t)=P(t)e^{Bt}.$$

This shows the first statement of the proposition.

We now evaluate $A^*+A=\text{blockdiag}(K_1^*+K_1,\ldots,K_r^*+K_r)$. Let us consider K_j as in the three cases above. If $m_j=1$, then K_j below does not contain the last sum with ϵ' in the following arguments, and the form of c_j is immediately clear.

Case 1: $\lambda_j\in\mathbb{R}^+$

$$K_j=\frac{1}{T}\left((\ln\lambda_j)I+\sum_{k=1}^{m_j-1}\frac{(-1)^{k+1}}{k}(\epsilon')^kN^k\right)\in\mathbb{R}^{m_j\times m_j};$$

hence, for $\mathbf{w}\in\mathbb{C}^{m_j}$

$$\begin{aligned}&\frac{1}{2}\mathbf{w}^*(K_j^*+K_j)\mathbf{w}\\ &= \frac{\ln\lambda_j}{T}\sum_{i=1}^{m_j}|w_i|^2\\ &\quad+\epsilon'\frac{1}{2T}\left(\overline{w_1}w_2+w_1\overline{w_2}+\overline{w_2}w_3+w_2\overline{w_3}+\ldots+\overline{w_{m_j-1}}w_{m_j}+w_{m_j-1}\overline{w_{m_j}}\right)\\ &\quad-\frac{(\epsilon')^2}{2}\frac{1}{2T}\left(\overline{w_1}w_3+w_1\overline{w_3}+\overline{w_2}w_4+w_2\overline{w_4}+\ldots+\overline{w_{m_j-2}}w_{m_j}+w_{m_j-2}\overline{w_{m_j}}\right)\\ &\quad+\ldots\\ &\quad+(-1)^{m_j}\frac{(\epsilon')^{m_j-1}}{m_j-1}\frac{1}{2T}\left(\overline{w_1}w_{m_j}+w_1\overline{w_{m_j}}\right).\end{aligned}$$

Note that with $\mathbb{R} \ni \overline{\xi}\eta + \xi\overline{\eta} \leq |\xi|^2 + |\eta|^2$ and $\epsilon' = \frac{1}{2}\min\left(\frac{\epsilon T}{2}, 1\right)$ we have

$$\begin{aligned}
\frac{1}{2}\mathbf{w}^*(K_j^* + K_j)\mathbf{w} &\leq \frac{\ln \lambda_j}{T}\sum_{i=1}^{m_j}|w_i|^2 \\
&\quad + \frac{\epsilon' + (\epsilon')^2 + \ldots + (\epsilon')^{m_j-1}}{T}\sum_{i=1}^{m_j}|w_i|^2 \\
&\leq \left(\frac{\ln \lambda_j}{T} + \epsilon\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)\right)\sum_{i=1}^{m_j}|w_i|^2 \\
&\leq \left(\frac{\ln \lambda_j}{T} + \epsilon\right)\sum_{i=1}^{m_j}|w_i|^2.
\end{aligned}$$

Case 2: $\lambda_j \in \mathbb{R}^-$

$$K_j = \frac{1}{T}\left((i\pi + \ln|\lambda_j|)I + \sum_{k=1}^{m_j-1}\frac{(-1)^{k+1}}{k}(-\epsilon')^k N^k\right) \in \mathbb{C}^{m_j \times m_j};$$

hence, for $\mathbf{w} \in \mathbb{C}^{m_j}$

$$\begin{aligned}
&\frac{1}{2}\mathbf{w}^*(K_j^* + K_j)\mathbf{w} \\
&= \frac{\ln|\lambda_j|}{T}\sum_{i=1}^{m_j}|w_i|^2 \\
&\quad -\epsilon'\frac{1}{2T}\left(\overline{w_1}w_2 + w_1\overline{w_2} + \overline{w_2}w_3 + w_2\overline{w_3} + \ldots + \overline{w_{m_j-1}}w_{m_j} + w_{m_j-1}\overline{w_{m_j}}\right) \\
&\quad -\frac{(\epsilon')^2}{2}\frac{1}{2T}\left(\overline{w_1}w_3 + w_1\overline{w_3} + \overline{w_2}w_4 + w_2\overline{w_4} + \ldots + \overline{w_{m_j-2}}w_{m_j} + w_{m_j-2}\overline{w_{m_j}}\right) \\
&\quad - \ldots \\
&\quad -\frac{(\epsilon')^{m_j-1}}{m_j-1}\frac{1}{2T}\left(\overline{w_1}w_{m_j} + w_1\overline{w_{m_j}}\right) \\
&\leq \left(\frac{\ln|\lambda_j|}{T} + \epsilon\right)\sum_{i=1}^{m_j}|w_i|^2
\end{aligned}$$

similarly to case 1.

Case 3: $\lambda_j = \alpha_j + i\beta_j$ **with** $\beta_j \neq 0$

Recall that

$$K_j = \frac{1}{T}\left((\ln r_j)I + \text{blockdiag}(\Theta, \Theta, \ldots, \Theta) + \sum_{k=1}^{2m_j-1}\frac{(-1)^{k+1}}{k}(\epsilon')^k\mathcal{N}^k\right) \in \mathbb{R}^{2m_j \times 2m_j};$$

where $\Theta = \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}$ and the nilpotent matrix $\mathcal{N}$ has 2×2 blocks of $\begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}$ on its super diagonal. Note that all entries of $\mathcal{N}^k$, $k \in \mathbb{N}$ are real and have an absolute value of ≤ 1 as they are of the form $\cos(k\theta_j)$ and $\pm\sin(k\theta_j)$ for $k = 1, 2, \ldots$.

Hence, for $\mathbf{w}\in\mathbb{C}^{2m_j}$

$$\begin{aligned}
\frac{1}{2}\mathbf{w}^*(K_j^*+K_j)\mathbf{w} &= \frac{\ln r_j}{T}\sum_{i=1}^{2m_j}|w_i|^2\\
&\quad +\epsilon'\frac{1}{2T}\Bigg(\cos\theta_j(\overline{w_1}w_3+w_1\overline{w_3})+\sin\theta_j(\overline{w_1}w_4+w_1\overline{w_4})\\
&\quad -\sin\theta_j(\overline{w_2}w_3+w_2\overline{w_3})+\cos\theta_j(\overline{w_2}w_4+w_2\overline{w_4})+\dots\Bigg)+\dots\\
&\leq \frac{\ln r_j}{T}\sum_{i=1}^{2m_j}|w_i|^2\\
&\quad +2\frac{\epsilon'+(\epsilon')^2+\ldots+(\epsilon')^{2m_j-1}}{T}\sum_{i=1}^{2m_j}|w_i|^2\\
&\leq \left(\frac{\ln r_j}{T}+\epsilon\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right)\right)\sum_{i=1}^{2m_j}|w_i|^2\\
&\leq \left(\frac{\ln r_j}{T}+\epsilon\right)\sum_{i=1}^{2m_j}|w_i|^2
\end{aligned}$$

since $\epsilon'=\frac{1}{2}\min\left(\frac{\epsilon T}{2},1\right)$. This shows the second statement of the proposition.

To show that $(P^{-1}(t))^*(S^{-1})^*S^{-1}P^{-1}(t)$ has real entries, note that

$$\begin{aligned}
P^{-1}(t) &= e^{Bt}\Phi^{-1}(t)\\
&= Se^{At}S^{-1}\Phi^{-1}(t)
\end{aligned}$$

so that

$$(P^{-1}(t))^*(S^{-1})^*S^{-1}P^{-1}(t) = (\Phi^{-1}(t))^*(S^{-1})^*(e^{At})^*e^{At}S^{-1}\Phi^{-1}(t).$$

It is thus sufficient to show that $(e^{At})^*e^{At}$ is real-valued, since all other matrices are real-valued. Note that since $A=\text{blockdiag}(K_1,\ldots,K_r)$, we have

$$\begin{aligned}
e^{At} &= \text{blockdiag}(e^{tK_1},\ldots,e^{tK_r}),\\
(e^{tA})^*e^{tA} &= \text{blockdiag}((e^{tK_1})^*e^{tK_1},\ldots,(e^{tK_r})^*e^{tK_r})
\end{aligned}$$

and the blocks where K_j have only real entries are trivially real-valued (cases 1 and 3). In case 2, $K_j=\frac{1}{T}((i\pi+\ln|\lambda_j|)I+N')$, where $N'\in\mathbb{R}^{m_j\times m_j}$ is a nilpotent, upper triangular matrix. Then, noting that I and N' commute,

$$\begin{aligned}
e^{tK_j} &= e^{\frac{t}{T}(i\pi+\ln|\lambda_j|)}\exp\left(\frac{t}{T}N'\right)\\
(e^{tK_j})^* &= e^{\frac{t}{T}(-i\pi+\ln|\lambda_j|)}\exp\left(\frac{t}{T}(N')^T\right)\\
(e^{tK_j})^*e^{tK_j} &= e^{\frac{2t}{T}\ln|\lambda_j|}\exp\left(\frac{t}{T}(N')^T\right)\exp\left(\frac{t}{T}N'\right),
\end{aligned}$$

which has real entries. □

Corollary 1. *Consider the ODE* $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ *with* $f\in C^\sigma(\mathbb{R}^n,\mathbb{R}^n)$, $\sigma\geq 2$ *and let* $S_t\mathbf{q}$ *be an exponentially stable periodic solution with period* T *and* $\mathbf{q}\in\mathbb{R}^n$. *Then the first variational equation* $\dot{\mathbf{y}}=D\mathbf{f}(S_t\mathbf{q})\mathbf{y}$ *is of the form as in the previous proposition* *with $s = \sigma - 1$; 1 is a single eigenvalue of $\Phi(T)$ with eigenvector $\mathbf{f}(\mathbf{q})$ and all other eigenvalues of $\Phi(T)$ satisfy $|\lambda| < 1$. More precisely, if $-\nu < 0$ is the maximal real part of all non-trivial Floquet exponents, we have $\frac{\ln|\lambda|}{T} \leq -\nu$. With the notations of Proposition 1 we can assume that $\lambda_1 = 1$ and $S\mathbf{e}_1 = \mathbf{f}(\mathbf{q})$.*

Then we have for all $\epsilon > 0$

$$\begin{aligned} \mathbf{f}(S_t\mathbf{q}) &= P(t)S\mathbf{e}_1 \text{ for all } t \in \mathbb{R} \\ \text{and } \frac{1}{2}\mathbf{w}^*(A^* + A)\mathbf{w} &\leq (-\nu + \epsilon)\left(\|\mathbf{w}\|^2 - |w_1|^2\right). \end{aligned}$$

for all $\mathbf{w} \in \mathbb{C}^n$, where $\|\mathbf{w}\| = \sqrt{\mathbf{w}^*\mathbf{w}}$.

Proof. Since $\mathbf{f}(S_t\mathbf{q})$ solves (5), we have $\mathbf{f}(S_t\mathbf{q}) = P(t)e^{Bt}\mathbf{f}(\mathbf{q})$ and, in particular for $t = T$, $\mathbf{f}(\mathbf{q}) = \mathbf{f}(S_T\mathbf{q}) = e^{BT}\mathbf{f}(\mathbf{q})$. Hence,

$$\begin{aligned} \mathbf{f}(S_t\mathbf{q}) &= P(t)Se^{At}S^{-1}\mathbf{f}(\mathbf{q}) \\ &= P(t)Se^{At}\mathbf{e}_1 \\ &= P(t)S\mathbf{e}_1 \end{aligned}$$

since $K_1 = 0$ in the definition of A. Proposition 1 shows the result, taking $\lambda_1 = 1$ and $m_1 = 1$ into account. □

3. **Converse theorem.** We will prove Theorem 1.3, showing that a contraction metric exists for an exponentially stable periodic orbit in the whole basin of attraction. Moreover, we can achieve the bound $-\nu + \epsilon$ for L_M for any fixed $\epsilon > 0$, where $-\nu$ denotes the largest real part of all non-trivial Floquet exponents.

Note that we consider contraction in directions $\mathbf{v}$ perpendicular to $\mathbf{f}(\mathbf{x})$ with respect to the metric M, i.e. $\mathbf{v}^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$. One could alternatively consider directions perpendicular to $\mathbf{f}(\mathbf{x})$ with respect to the Euclidean metric, i.e. $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$, but then the function L_M needs to reflect this, see [5, 1].

In the proof we will first construct $M = M_0$ on the periodic orbit Ω using Floquet theory. Then, we define a projection π of points in a neighborhood U of Ω onto Ω such that $(\mathbf{x} - \pi(\mathbf{x}))^T M_0(\pi(\mathbf{x}))\mathbf{f}(\pi(\mathbf{x})) = 0$, which will be used to synchronize the time of solutions such that $\pi(S_\tau\mathbf{x}) = S_{\theta_{\mathbf{x}}(\tau)}\pi(\mathbf{x})$. Finally, M will be defined through a scalar-valued function V by $M(\mathbf{x}) = M_1(\mathbf{x})e^{2V(\mathbf{x})}$, where $M_1 = M_0$ on the periodic orbit.

Proof. (of Theorem 1.3) Note that we assume $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ to achieve more detailed results concerning the smoothness and assume lower bounds on σ as appropriate for each result; we always assume at least $\sigma \geq 2$.

I. Definition and properties of M_0 on Ω

We fix a point $\mathbf{q} \in \Omega$ and consider the first variational equation

$$\dot{\mathbf{y}} = D\mathbf{f}(S_t\mathbf{q})\mathbf{y} \tag{10}$$

which is a T-periodic, linear equation for $\mathbf{y}$, and $D\mathbf{f} \in C^{\sigma-1}$. By Proposition 1 and Corollary 1 the principal fundamental matrix solution $\Phi \in C^{\sigma-1}(\mathbb{R}, \mathbb{R}^{n\times n})$ of (10) with $\Phi(0) = I$ can be written as

$$\Phi(t) = P(S_t\mathbf{q})e^{Bt},$$

where $B \in \mathbb{C}^{n\times n}$; note that $P \in C^{\sigma-1}(\mathbb{R}, \mathbb{C}^{n\times n})$ can be defined on the periodic orbit as it is T-periodic. By the assumptions on Ω, the eigenvalues of B are 0 with algebraic multiplicity one and the others have a real part $\leq -\nu < 0$.

We define S as in Proposition 1 and define the $C^{\sigma-1}$-function

$$M_0(S_t\mathbf{q}) \quad = \quad P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q}) \in \mathbb{R}^{n\times n}. \tag{11}$$

Note that $M_0(S_t\mathbf{q})$ is real by Proposition 1, symmetric, since it is Hermitian and real, and positive definite by

$$\mathbf{v}^T M_0(S_t\mathbf{q})\mathbf{v} \quad = \quad \|S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{v}\|^2 \text{ for all } \mathbf{v} \in \mathbb{R}^n \tag{12}$$

and since $S^{-1}P^{-1}(S_t\mathbf{q})$ is non-singular.

We will now calculate $L_{M_0}(S_t\mathbf{q};\mathbf{v})$. First, we have for the orbital derivative

$$M_0'(S_t\mathbf{q}) = ((P^{-1}(S_t\mathbf{q}))')^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q}) + P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}(P^{-1}(S_t\mathbf{q}))'.$$

Furthermore, by using $(P^{-1}(S_t\mathbf{q})P(S_t\mathbf{q}))' = 0$, we obtain

$$(P^{-1}(S_t\mathbf{q}))' = -P^{-1}(S_t\mathbf{q})(P(S_t\mathbf{q}))'P^{-1}(S_t\mathbf{q}).$$

In addition, since $t \mapsto P(S_t\mathbf{q})e^{Bt}$ is a solution of (10), we have $(P(S_t\mathbf{q}))' = D\mathbf{f}(S_t\mathbf{q})P(S_t\mathbf{q}) - P(S_t\mathbf{q})B$. Altogether, we get

$$(P^{-1}(S_t\mathbf{q}))' \quad = \quad -P^{-1}(S_t\mathbf{q})D\mathbf{f}(S_t\mathbf{q}) + BP^{-1}(S_t\mathbf{q}). \tag{13}$$

Hence,

$$\begin{aligned} M_0'(S_t\mathbf{q}) = & -D\mathbf{f}(S_t\mathbf{q})^T P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q}) \\ & + P^{-1}(S_t\mathbf{q})^*B^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q}) \\ & - P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q})D\mathbf{f}(S_t\mathbf{q}) \\ & + P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}BP^{-1}(S_t\mathbf{q}). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & M_0(S_t\mathbf{q})D\mathbf{f}(S_t\mathbf{q}) + D\mathbf{f}(S_t\mathbf{q})^T M_0(S_t\mathbf{q}) + M_0'(S_t\mathbf{q}) \\ & \quad = \quad P^{-1}(S_t\mathbf{q})^*B^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q}) + P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}BP^{-1}(S_t\mathbf{q}). \end{aligned}$$

Furthermore, we have for $\mathbf{v} \in \mathbb{R}^n$

$$\begin{aligned} L_{M_0}(S_t\mathbf{q};\mathbf{v}) \quad & = \quad \frac{1}{2}\mathbf{v}^T\left(M_0(S_t\mathbf{q})D\mathbf{f}(S_t\mathbf{q}) + D\mathbf{f}(S_t\mathbf{q})^T M_0(S_t\mathbf{q}) + M_0'(S_t\mathbf{q})\right)\mathbf{v} \\ & = \quad \mathbf{v}^T P^{-1}(S_t\mathbf{q})^*(S^{-1})^*\left(\frac{1}{2}\left(S^*B^*(S^{-1})^* + S^{-1}BS\right)\right)S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{v} \\ & = \quad \mathbf{w}^*\left(\frac{1}{2}\left(A^* + A\right)\right)\mathbf{w}, \end{aligned} \tag{14}$$

where $\mathbf{w} := S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{v} \in \mathbb{C}^n$ and $A = S^{-1}BS$.

For $\mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v}^T M_0(S_t\mathbf{q})\mathbf{v} = 1$ and $\mathbf{f}(S_t\mathbf{q})^T M_0(S_t\mathbf{q})\mathbf{v} = 0$ we have

$$\begin{aligned} \mathbf{w}^*\mathbf{w} \quad & = \quad \mathbf{v}^T P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{v} \\ & = \quad \mathbf{v}^T M_0(S_t\mathbf{q})\mathbf{v} \\ & = \quad 1 \end{aligned}$$

and, using $\mathbf{e}_1 = S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{f}(S_t\mathbf{q})$ from Corollary 1

$$\begin{aligned} w_1 \quad & = \quad \mathbf{e}_1^*\mathbf{w} \\ & = \quad \mathbf{f}(S_t\mathbf{q})^T P^{-1}(S_t\mathbf{q})^*(S^{-1})^*S^{-1}P^{-1}(S_t\mathbf{q})\mathbf{v} \\ & = \quad \mathbf{f}(S_t\mathbf{q})^T M_0(S_t\mathbf{q})\mathbf{v} \\ & = \quad 0. \end{aligned}$$

This shows with Corollary 1 and (14)

$$\begin{aligned} L_{M_0}(S_t\mathbf{q}) &= \max_{\mathbf{v}^T M_0(S_t\mathbf{q})\mathbf{v}=1, \mathbf{v}^T M_0(S_t\mathbf{q})\mathbf{f}(S_t\mathbf{q})=0} L_{M_0}(S_t\mathbf{q};\mathbf{v}) \\ &\leq \max_{\mathbf{w}\in\mathbb{C}^n, w_1=0, \|\mathbf{w}\|=1} (-\nu+\epsilon)(\|\mathbf{w}\|^2 - |w_1|^2) \\ &\leq -\nu+\epsilon. \end{aligned} \quad (15)$$

II. Projection

Fix a point $\mathbf{q}\in\Omega$ on the periodic orbit. For $\mathbf{x}$ near the periodic orbit we define the projection $\pi(\mathbf{x}) = S_\theta\mathbf{q}$ on the periodic orbit orthogonal to $\mathbf{f}(S_\theta\mathbf{q})$ with respect to the scalar product $\langle \mathbf{v},\mathbf{w}\rangle_{M_0(S_\theta\mathbf{q})} = \mathbf{v}^T M_0(S_\theta\mathbf{q})\mathbf{w}$ implicitly by (16) below. The following lemma is based on the implicit function theorem and shows that the projection can be defined in a neighborhood of the periodic orbit, not just locally.

Lemma 3.1. *Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{f}\in C^\sigma(\mathbb{R}^n,\mathbb{R}^n)$ with $\sigma\geq 2$.*

Then there is a compact, positively invariant neighborhood U of Ω with $U\subset A(\Omega)$ and a function $\pi\in C^{\sigma-1}(U,\Omega)$ such that $\pi(\mathbf{x})=\mathbf{x}$ if and only if $\mathbf{x}\in\Omega$. Moreover, for all $\mathbf{x}\in U$ we have

$$(\mathbf{x}-\pi(\mathbf{x}))^T M_0(\pi(\mathbf{x}))\mathbf{f}(\pi(\mathbf{x})) = 0. \quad (16)$$

Proof. Fix a point $\mathbf{q}\in\Omega$ and define M_0 by (11). Define the $C^{\sigma-1}$ function

$$G(\mathbf{x},\theta) = (\mathbf{x}-S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q})$$

for $\mathbf{x}\in\mathbb{R}^n$, $\theta\in\mathbb{R}$.

Define the following constants:

$$\begin{aligned} \min_{\mathbf{p}\in\Omega}\|\mathbf{f}(\mathbf{p})\| &= c_1 > 0 \\ \max_{\mathbf{p}\in\Omega}\|\mathbf{f}(\mathbf{p})\| &= c_2 > 0 \\ \max_{\mathbf{p}\in\Omega}\|D\mathbf{f}(\mathbf{p})\| &= c_3 \\ \max_{\mathbf{p}\in\Omega}\|P(\mathbf{p})\| &= p_1 \\ \max_{\mathbf{p}\in\Omega}\|P^{-1}(\mathbf{p})\| &= p_2 \\ \min_{\mathbf{p}\in\Omega}\|M_0(\mathbf{p})\| &= m_1 > 0 \\ \max_{\mathbf{p}\in\Omega}\|M_0(\mathbf{p})\| &= m_2 \\ \max_{\mathbf{p}\in\Omega}\|M_0'(\mathbf{p})\| &= m_3, \end{aligned}$$

with the matrix norm $\|\cdot\| = \|\cdot\|_2$, which is induced by the vector norm $\|\cdot\| = \|\cdot\|_2$ and is sub-multiplicative. We will first prove the following quantitative version of the local implicit function theorem, using that θ is one-dimensional.

Lemma 3.2. *There are constants $\delta,\epsilon>0$ such that for each point $\mathbf{x}_0 = S_{\theta_0}\mathbf{q}\in\Omega$, there is a function $p_{\mathbf{x}_0}\in C^{\sigma-1}(B_\delta(\mathbf{x}_0), B_\epsilon(\theta_0))$ such that for all $(\mathbf{x},\theta)\in B_\delta(\mathbf{x}_0)\times B_\epsilon(\theta_0)$*

$$G(\mathbf{x},\theta) = 0 \Longleftrightarrow \theta = p_{\mathbf{x}_0}(\mathbf{x}).$$

If $\mathbf{x}\in B_{\delta/2}(\mathbf{x}_0)$, then $p_{\mathbf{x}_0}(\mathbf{x})\in B_{\epsilon/2}(\theta_0)$.

Proof. We have

$$
\begin{aligned}
G_\theta(\mathbf{x},\theta) &= \frac{d}{d\theta}(\mathbf{x}-S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q})\\
&= -\mathbf{f}(S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q})\\
&\quad +(\mathbf{x}-S_\theta\mathbf{q})^T M_0'(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q})\\
&\quad +(\mathbf{x}-S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q})D\mathbf{f}(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q})\,.
\end{aligned}
$$

With $\min_{\theta\in[0,T]}\mathbf{f}(S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q})\mathbf{f}(S_\theta\mathbf{q}) \geq c_1^2 m_1 > 0$ we have for all $\|\mathbf{x}-S_\theta\mathbf{q}\| < \delta_2 := \frac{c_1^2 m_1}{2c_2(m_3+m_2c_3)}$

$$
G_\theta(\mathbf{x},\theta) \quad < \quad -c_1^2 m_1 + \delta_2 c_2(m_3+m_2c_3) \;=\; -\frac{c_1^2 m_1}{2} < 0.
$$

Let $\delta_1 := \frac{\delta_2}{2}$ and $\epsilon_1 := \frac{\delta_2}{2c_2}$. For any $\mathbf{x}_0 = S_{\theta_0}\mathbf{q} \in \Omega$ we have for all $\mathbf{x}\in\mathbb{R}^n$ with $\|\mathbf{x}-\mathbf{x}_0\| < \delta_1$ and all $\theta\in\mathbb{R}$ with $|\theta-\theta_0| < \epsilon_1$

$$
G_\theta(\mathbf{x},\theta) \quad < \quad -\frac{c_1^2 m_1}{2} < 0 \tag{17}
$$

since

$$
\|\mathbf{x}-S_\theta\mathbf{q}\| \leq \|\mathbf{x}-\mathbf{x}_0\| + \|S_{\theta_0}\mathbf{q}-S_\theta\mathbf{q}\| < \delta_1 + |\theta_0-\theta|c_2 < \delta_2.
$$

Since $G(\mathbf{x}_0,\theta_0)=0$ we have with $\epsilon := \epsilon_1/2$ by (17)

$$
\begin{aligned}
G(\mathbf{x}_0,\theta_0+\epsilon) &< -\frac{c_1^2 m_1}{2}\epsilon,\\
G(\mathbf{x}_0,\theta_0-\epsilon) &> \frac{c_1^2 m_1}{2}\epsilon.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\nabla_{\mathbf{x}} G(\mathbf{x},\theta) &= \mathbf{f}(S_\theta\mathbf{q})^T M_0(S_\theta\mathbf{q}),\\
\|\nabla_{\mathbf{x}} G(\mathbf{x},\theta)\| &\leq c_2 m_2
\end{aligned}
$$

for all $\mathbf{x}\in\mathbb{R}^n$ and $\theta\in\mathbb{R}$.

Define $\delta := \min\left(\delta_1, \frac{c_1^2 m_1}{4c_2m_2}\epsilon\right)$. For $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ we have

$$
\begin{aligned}
G(\mathbf{x},\theta_0+\epsilon) &\leq G(\mathbf{x}_0,\theta_0+\epsilon)\\
&\quad + \int_0^1 \nabla_{\mathbf{x}} G(\mathbf{x}_0+\lambda(\mathbf{x}-\mathbf{x}_0),\theta_0+\epsilon)\,d\lambda\cdot(\mathbf{x}-\mathbf{x}_0)\\
&< -\frac{c_1^2 m_1}{2}\epsilon + c_2 m_2\delta\\
&\leq -\frac{c_1^2 m_1}{4}\epsilon \;<\; 0\\
\text{and similarly } G(\mathbf{x},\theta_0-\epsilon) &> \frac{c_1^2 m_1}{4}\epsilon \;>\; 0.
\end{aligned}
$$

Since $G(\mathbf{x},\theta)$ is strictly decreasing with respect to θ in $B_\epsilon(\theta_0)$ by (17), the intermediate value theorem implies that there is a unique $\theta^*\in(\theta_0-\epsilon,\theta_0+\epsilon)$ such that $G(\mathbf{x},\theta^*)=0$, which defines a function $p_{\mathbf{x}_0}(\mathbf{x})=\theta^*$. The statement for $\epsilon/2$ and $\delta/2$ follows similarly. The smoothness of $p_{\mathbf{x}_0}$ follows by the classical implicit function theorem, since $G\in C^{\sigma-1}$. □

Now we want to define π by using the $p_{\mathbf{x}}$ for finitely many points $\mathbf{x} \in \Omega$. Denote the (minimal) period of the periodic orbit by T; we can assume that $\epsilon < T$. Define

$$c := \min_{\mathbf{p}\in\Omega} \min_{\theta\in[-T/2,T/2]\setminus(-\epsilon/2,\epsilon/2)} \|S_\theta \mathbf{p} - \mathbf{p}\| > 0.$$

We can conclude that if $\|S_\theta \mathbf{p} - \mathbf{p}\| \leq c/2$ with $\mathbf{p} \in \Omega$ and $|\theta| \leq T/2$, then $|\theta| < \epsilon/2$.

Let $\delta' = \min(\delta/2, c/4)$. Since Ω is compact and $\Omega \subset \bigcup_{\mathbf{x}_0\in\Omega} B_{\delta'}(\mathbf{x}_0)$, there is a finite number of $\mathbf{x}_i = S_{\theta_i}\mathbf{q} \in \Omega$, $i = 1\ldots, N$, with

$$\Omega \quad \subset \quad \bigcup_{i=1}^N B_{\delta'}(\mathbf{x}_i) =: \widetilde{U}, \tag{18}$$

such that $\widetilde{U}$ is an open neighborhood of Ω. We want to show that the $p_{\mathbf{x}_i} = p_i$ define a unique function $p\colon \widetilde{U} \to S^1_T$, where S^1_T are the reals modulo T such that $p = p_i$ on $B_{\delta'}(\mathbf{x}_i)$. We need to show that if $\mathbf{x} \in B_{\delta'}(\mathbf{x}_i) \cap B_{\delta'}(\mathbf{x}_j)$, then $p_i(\mathbf{x}) = p_j(\mathbf{x})$.

Let $\mathbf{x} \in B_{\delta'}(\mathbf{x}_i) \cap B_{\delta'}(\mathbf{x}_j)$ and, without loss of generality, $|\theta_j - \theta_i| \leq T/2$ since the θ_i and θ_j are uniquely defined only modulo T. Then

$$\begin{aligned} \|\mathbf{x}_i - S_{\theta_j-\theta_i}\mathbf{x}_i\| &= \|\mathbf{x}_i - \mathbf{x}_j\| \\ &\leq \|\mathbf{x}_i - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_j\| \\ &< 2\delta' \leq \min(\delta, c/2). \end{aligned}$$

Hence, $|\theta_j - \theta_i| < \epsilon/2$.

Since $\mathbf{x} \in B_{\delta/2}(\mathbf{x}_i) \cap B_{\delta/2}(\mathbf{x}_j)$, we have $p_i(\mathbf{x}) \in B_{\epsilon/2}(\theta_i)$ and $p_j(\mathbf{x}) \in B_{\epsilon/2}(\theta_j)$ by Lemma 3.2. Then

$$|p_i(\mathbf{x}) - \theta_j| \quad \leq \quad |p_i(\mathbf{x}) - \theta_i| + |\theta_i - \theta_j| \ < \ \epsilon$$

and similarly $p_j(\mathbf{x}) \in B_\epsilon(\theta_i)$. Moreover, $\mathbf{x} \in B_\delta(\mathbf{x}_j) \cap B_\delta(\mathbf{x}_j)$. Lemma 3.2 implies that $\theta = p_i(\mathbf{x})$ if and only if $G(\mathbf{x}, \theta) = 0$ if and only if $\theta = p_j(\mathbf{x})$, which shows $p_i(\mathbf{x}) = p_j(\mathbf{x})$.

Since Ω is stable, we can choose $\Omega \subset U^\circ \subset U \subset \widetilde{U}$ such that U is compact and positively invariant. For $\mathbf{x} \in U$ define $\pi(\mathbf{x}) = S_{p(\mathbf{x})}\mathbf{q}$. Since p is defined by $p_{\mathbf{x}_i}$, we have by Lemma 3.2 that $0 = G(\mathbf{x}, p(\mathbf{x})) = (\mathbf{x} - \pi(\mathbf{x}))^T M_0(\pi(\mathbf{x}))\mathbf{f}(\pi(\mathbf{x}))$.

If $\mathbf{x} = S_\theta \mathbf{q} \in \Omega$, then there is a $\mathbf{x}_i = S_{\theta_i}\mathbf{q} \in \Omega$ by (18) such that $\mathbf{x} \in B_{\delta'}(\mathbf{x}_i)$ and thus, as above, $|\theta - \theta_i| < \epsilon/2$. Hence, by Lemma 3.2, as $\mathbf{x} \in B_\delta(\mathbf{x}_i)$ and $\theta \in B_\epsilon(\theta_i)$, $p_i(\mathbf{x}) = \theta$ and thus $\pi(\mathbf{x}) = \mathbf{x}$, as this satisfies $0 = G(\mathbf{x}, \theta)$. If $\mathbf{x} \notin \Omega$, then, since $\pi(\mathbf{x}) \in \Omega$, $\mathbf{x} \neq \pi(\mathbf{x})$. This shows the lemma. $\square$

III. Synchronization

In this step we synchronize the time between the solution $S_t\mathbf{x}$ and the solution on the periodic orbit $S_\theta \pi(\mathbf{x})$ such that (19) holds. This will enable us later to define a distance between $S_t\mathbf{x}$ and Ω in Step IV.

Definition 3.3. For $\mathbf{x} \in U$ we can define $\theta_{\mathbf{x}} \in C^{\sigma-1}(\mathbb{R}^+_0, \mathbb{R})$ by $\theta_{\mathbf{x}}(0) = 0$ and

$$S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}) \quad = \quad \pi(S_t\mathbf{x}) \tag{19}$$

for all $t \geq 0$.

We have

$$\begin{aligned}\dot{\theta}_{\mathbf{x}}(t) &= (\mathbf{f}(S_t\mathbf{x})^T M_0(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))) \\ &\quad \Big(\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))^T M_0(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})) \\ &\quad -(S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))^T \big[M_0'(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})) \\ &\quad\quad +M_0(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))D\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\big]\Big)^{-1}. \qquad (20)\end{aligned}$$

The denominator of (20) is strictly positive for all $t \geq 0$ and $\mathbf{x} \in U$.

Proof. Denote $\pi(\mathbf{x}) =: \mathbf{p} \in \Omega$. Observe, that both sides of (19) equal for $t = 0$. For any $t \geq 0$, we have $S_t\mathbf{x} \in U$, and since $\pi(S_t\mathbf{x})$ denotes a point on the periodic orbit, we can write it as $\pi(S_t\mathbf{x}) = S_{\theta_{\mathbf{x}}(t)}\mathbf{p}$. Note that $\theta_{\mathbf{x}}(t)$ is only uniquely defined modulo T, however, it is uniquely defined by the requirement that $\theta_{\mathbf{x}}$ is a continuous function.

By (16), we have

$$(S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\mathbf{p})^T M_0(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p}) = 0.$$

Hence, $\theta_{\mathbf{x}}(t)$ is implicitly defined by

$$Q(t,\theta) = (S_t\mathbf{x} - S_\theta\mathbf{p})^T M_0(S_\theta\mathbf{p})\mathbf{f}(S_\theta\mathbf{p}) = 0. \qquad (21)$$

Note that $\theta_{\mathbf{x}} \in C^{\sigma-1}(\mathbb{R}_0^+, \mathbb{R})$ by the Implicit Function Theorem which implies

$$\begin{aligned}\frac{d\theta_{\mathbf{x}}}{dt} &= -\frac{\partial_t Q(t,\theta)}{\partial_\theta Q(t,\theta)}\bigg|_{\theta=\theta_{\mathbf{x}}(t)} \\ &= (\mathbf{f}(S_t\mathbf{x})^T M_0(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})) \\ &\quad \Big(\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})^T M_0(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p}) \\ &\quad -(S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\mathbf{p})^T M_0'(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p}) \\ &\quad -(S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\mathbf{p})^T M_0(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})D\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\mathbf{f}(S_{\theta_{\mathbf{x}}(t)}\mathbf{p})\Big)^{-1}.\end{aligned}$$

With the notations of the proof of Lemma 3.1, for $S_t\mathbf{x} \in U$ there is a point $\mathbf{x}_i = S_{\theta_i}\mathbf{q} \in \Omega$ such that $S_t\mathbf{x} \in B_{\delta'}(\mathbf{x}_i)$. We have $S_t\mathbf{x} \in B_\delta(\mathbf{x}_i)$ and, modulo T, we have $p_i(S_t\mathbf{x}) = \theta_{\mathbf{x}}(t) \in B_\epsilon(\theta_i)$. Hence, the denominator is $> \frac{c_1^2 m_1}{2}$ by (17). ☐

Lemma 3.4. *For $\mathbf{x} \in U$ we have*

$$S_{\theta_{S_\tau\mathbf{x}}(t)}\pi(S_\tau\mathbf{x}) = S_{\theta_{\mathbf{x}}(t+\tau)}\pi(\mathbf{x})$$

for all $t, \tau \geq 0$.

Proof. We apply (19) to the point $S_\tau\mathbf{x}$ and the time t, obtaining

$$S_{\theta_{S_\tau\mathbf{x}}(t)}\pi(S_\tau\mathbf{x}) = \pi(S_tS_\tau\mathbf{x}).$$

Now we apply (19) to the point $\mathbf{x}$ and the time $t+\tau$, obtaining

$$S_{\theta_{\mathbf{x}}(t+\tau)}\pi(\mathbf{x}) = \pi(S_{t+\tau}\mathbf{x}).$$

As both right-hand sides are equal by the semi-flow property, this proves the statement. ☐

IV. Distance to the periodic orbit

In the following lemma we define a distance of points in U to the periodic orbit, and we show that it decreases exponentially. Note that the ϵ in the following lemma is not related to the ϵ in the previous steps.

Lemma 3.5. *Let $\epsilon < \min(1, \nu/2)$ and $\sigma \geq 2$. Then there is a positively invariant, compact neighborhood U of the periodic orbit Ω such that the function $d \in C^{\sigma-1}(U, \mathbb{R}_0^+)$, defined by*

$$d(\mathbf{x}) = (\mathbf{x} - \pi(\mathbf{x}))^T M_0(\pi(\mathbf{x}))(\mathbf{x} - \pi(\mathbf{x}))$$

satisfies $d(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \Omega$. Moreover, $d'(\mathbf{x}) < 0$ for all $\mathbf{x} \in U \setminus \Omega$ and

$$\begin{array}{rcl} d(S_t\mathbf{x}) & \leq & e^{2(-\nu+2\epsilon)t} d(\mathbf{x}) \text{ for all } \mathbf{x} \in U \text{ and all } t \geq 0, \\ 1-\epsilon & \leq & \dot{\theta}_{\mathbf{x}}(t) \ \leq \ 1+\epsilon \text{ for all } t \geq 0. \end{array}$$

Proof. Note that d is $C^{\sigma-1}$ as all of its terms are. As $M_0(\mathbf{x})$ is positive definite, $d(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \pi(\mathbf{x})$, i.e. $\mathbf{x} \in \Omega$ by Lemma 3.1. Define

$$c_4 := \frac{\epsilon}{2p_1\, p_2 \|S^{-1}\|\, \|S\|} > 0, \tag{22}$$

$$c_5 = 2c_2 \frac{2c_3 m_2 + m_3 + c_4 m_2}{c_1^2 m_1}, \tag{23}$$

$$c_6 := \frac{\epsilon}{2p_1\, p_2 \|S^{-1}\|\, \|S\|\, c_5(c_3 + \|B\|)}. \tag{24}$$

where the constants where defined in Step II, proof of Lemma 3.1.

For $\mathbf{y} \in U$ we use the Taylor expansion around $\pi(\mathbf{y}) \in \Omega$. Hence, there is a function $\boldsymbol{\psi}(\mathbf{y})$ satisfying

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}(\pi(\mathbf{y})) + D\mathbf{f}(\pi(\mathbf{y}))(\mathbf{y} - \pi(\mathbf{y})) + \boldsymbol{\psi}(\mathbf{y}) \tag{25}$$

with $\|\boldsymbol{\psi}(\mathbf{y})\| \leq c_4 \|\mathbf{y} - \pi(\mathbf{y})\|$ for all $\mathbf{y} \in U$, noting that Ω is compact, where we choose U still to be a positively invariant, compact neighborhood of Ω, possibly smaller than before and such that also have

$$\|\mathbf{y} - \pi(\mathbf{y})\| \leq \delta_0 := \min\left(c_6, \frac{c_1^2 m_1}{2c_2[m_3 + m_2 c_3]}, \frac{\epsilon}{c_5}, 1\right) \text{ for all } \mathbf{y} \in U. \tag{26}$$

Recall that, due to the definition of M_0 and (19) we have

$$\begin{array}{rcl} d(\mathbf{x}) & = & (\mathbf{x} - \pi(\mathbf{x}))^T (P^{-1}(\pi(\mathbf{x})))^* (S^{-1})^* S^{-1} P^{-1}(\pi(\mathbf{x}))(\mathbf{x} - \pi(\mathbf{x})) \\ d(S_t\mathbf{x}) & = & (S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))^T (P^{-1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})))^* (S^{-1})^* \\ & & S^{-1} P^{-1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))(S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})). \end{array}$$

Now let us calculate the orbital derivative, denoting $\theta(t) := \theta_{\mathbf{x}}(t)$.

$$\begin{aligned} d'(S_t\mathbf{x}) &= \Bigg[\frac{d}{dt}\left(P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))\right)(S_t\mathbf{x} - S_{\theta(t)}\pi(\mathbf{x})) \\ &\quad + P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))[\mathbf{f}(S_t\mathbf{x}) - \dot\theta(t)\mathbf{f}(S_{\theta(t)}\pi(\mathbf{x}))]\Bigg]^* \\ &\quad (S^{-1})^* S^{-1} P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))(S_t\mathbf{x} - S_{\theta(t)}\pi(\mathbf{x})) \\ &\quad + (S_t\mathbf{x} - S_{\theta(t)}\pi(\mathbf{x}))^T (P^{-1}(S_{\theta(t)}\pi(\mathbf{x})))^* (S^{-1})^* S^{-1} \\ &\quad \Bigg[\frac{d}{dt}\left(P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))\right)(S_t\mathbf{x} - S_{\theta(t)}\pi(\mathbf{x})) \\ &\quad + P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))[\mathbf{f}(S_t\mathbf{x}) - \dot\theta(t)\mathbf{f}(S_{\theta(t)}\pi(\mathbf{x}))]\Bigg]. \end{aligned}$$

We denote $\mathbf{p} := \pi(\mathbf{x})$ and $\mathbf{v}(t) := S_t\mathbf{x} - S_{\theta(t)}\pi(\mathbf{x}) = S_t\mathbf{x} - \pi(S_t\mathbf{x})$ by (19). Hence, using (26) for $\mathbf{y} = S_t\mathbf{x} \in U$ since $\mathbf{x} \in U$, which is positively invariant, we have

$$\|\mathbf{v}(t)\| \leq \delta_0 = \min\left(c_6, \frac{c_1^2 m_1}{2c_2[m_3 + m_2 c_3]}, \frac{\epsilon}{c_5}, 1\right) \tag{27}$$

for all $t \geq 0$. We have

$$\frac{d}{dt}\left(P^{-1}(S_{\theta(t)}\pi(\mathbf{x}))\right) = \dot\theta(t)(-P^{-1}(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p}) + BP^{-1}(S_{\theta(t)}\mathbf{p}))$$

by (13). Thus,

$$\begin{aligned} d'(S_t\mathbf{x}) &= \Bigg[\dot\theta(t)(-P^{-1}(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p}) + BP^{-1}(S_{\theta(t)}\mathbf{p}))\mathbf{v}(t) \\ &\quad + P^{-1}(S_{\theta(t)}\mathbf{p})[\mathbf{f}(S_t\mathbf{x}) - \dot\theta(t)\mathbf{f}(S_{\theta(t)}\mathbf{p})]\Bigg]^* \\ &\quad (S^{-1})^* S^{-1} P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) \\ &\quad + \mathbf{v}(t)^T (P^{-1}(S_{\theta(t)}\mathbf{p}))^* (S^{-1})^* S^{-1} \\ &\quad \Bigg[\dot\theta(t)(-P^{-1}(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p}) + BP^{-1}(S_{\theta(t)}\mathbf{p}))\mathbf{v}(t) \\ &\quad + P^{-1}(S_{\theta(t)}\mathbf{p})[\mathbf{f}(S_t\mathbf{x}) - \dot\theta(t)\mathbf{f}(S_{\theta(t)}\mathbf{p})]\Bigg]. \end{aligned} \tag{28}$$

Using the Taylor expansion (25) for $\mathbf{y} = S_t\mathbf{x}$, we obtain with $\pi(S_t\mathbf{x}) = S_{\theta(t)}\mathbf{p}$,

$$\mathbf{f}(S_t\mathbf{x}) = \mathbf{f}(S_{\theta(t)}\mathbf{p}) + D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) + \boldsymbol{\psi}(S_t\mathbf{x}) \tag{29}$$

and thus with (20)

$$\begin{aligned}
&\dot\theta(t)-1\\
&= \Big(\mathbf{f}(S_t\mathbf{x})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) - \mathbf{f}(S_{\theta(t)}\mathbf{p})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\\
&\qquad + \mathbf{v}(t)^T M_0'(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) + \mathbf{v}(t)^T M_0(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\Big)\\
&\quad \Big(\mathbf{f}(S_{\theta(t)}\mathbf{p})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) - \mathbf{v}(t)^T M_0'(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\\
&\qquad - \mathbf{v}(t)^T M_0(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\Big)^{-1}\\
&= \Big(\mathbf{v}(t)^T D\mathbf{f}(S_{\theta(t)}\mathbf{p})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) + \boldsymbol{\psi}(S_t\mathbf{x})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\\
&\quad + \quad \mathbf{v}(t)^T M_0'(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) + \mathbf{v}(t)^T M_0(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\Big)\\
&\quad \Big(\mathbf{f}(S_{\theta(t)}\mathbf{p})^T M_0(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) - \mathbf{v}(t)^T M_0'(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\\
&\qquad - \mathbf{v}(t)^T M_0(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\Big)^{-1}
\end{aligned}$$

which shows, using (27) and (23),

$$\begin{aligned}
|\dot\theta(t)-1| &\leq \frac{\|\mathbf{v}(t)\|c_2[2c_3m_2+m_3] + \|\boldsymbol{\psi}(S_t\mathbf{x})\|m_2c_2}{c_1^2m_1 - \|\mathbf{v}(t)\|c_2[m_3+m_2c_3]}\\
&\leq 2c_2\frac{2c_3m_2+m_3+c_4m_2}{c_1^2m_1}\|\mathbf{v}(t)\| = c_5\|\mathbf{v}(t)\| \leq \epsilon\,. \qquad (30)
\end{aligned}$$

In particular, we have $1-\epsilon \leq \dot\theta(t) \leq 1+\epsilon$, which shows the existence of $\theta(t)$ for all $t \geq 0$, $\dot\theta(t) > 0$ for all $t \geq 0$, that $\theta(t)$ is a bijective function from $[0,\infty)$ to $[0,\infty)$ and $\lim_{t\to\infty}\theta(t) = \infty$.

Hence, we have from (28) and (29)

$$\begin{aligned}
d'(S_t\mathbf{x}) &= \big[(1-\dot\theta(t))P^{-1}(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) + BP^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)\\
&\qquad -(1-\dot\theta(t))BP^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) + (1-\dot\theta(t))P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p})\\
&\qquad +P^{-1}(S_{\theta(t)}\mathbf{p})\boldsymbol{\psi}(S_t\mathbf{x})\big]^*(S^{-1})^*S^{-1}P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)\\
&\quad +\mathbf{v}(t)^T(P^{-1}(S_{\theta(t)}\mathbf{p}))^*(S^{-1})^*S^{-1}\big[(1-\dot\theta(t))P^{-1}(S_{\theta(t)}\mathbf{p})D\mathbf{f}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)\\
&\qquad +BP^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) - (1-\dot\theta(t))BP^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)\\
&\qquad +(1-\dot\theta(t))P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{f}(S_{\theta(t)}\mathbf{p}) + P^{-1}(S_{\theta(t)}\mathbf{p})\boldsymbol{\psi}(S_t\mathbf{x})\big]\\
&\leq 2\|S^{-1}P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)\|\,\|S^{-1}\|\,\|P^{-1}(S_{\theta(t)}\mathbf{p})\|\\
&\qquad \Big[|1-\dot\theta(t)|(\|D\mathbf{f}(S_{\theta(t)}\mathbf{p})\| + \|B\|)\|\mathbf{v}(t)\| + \|\boldsymbol{\psi}(S_t\mathbf{x})\|\Big]\\
&\quad +\mathbf{v}(t)^*(P^{-1}(S_{\theta(t)}\mathbf{p}))^*\left[(S^{-1})^*S^{-1}B + B^*(S^{-1})^*S^{-1}\right]P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)
\end{aligned}$$

using

$$0 = \mathbf{f}(S_{\theta(t)}\mathbf{p})^*M_0(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) = \mathbf{f}(S_{\theta(t)}\mathbf{p})^*(P^{-1}(S_{\theta(t)}\mathbf{p}))^*(S^{-1})^*S^{-1}P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t)$$

by (21).

Setting $\mathbf{w}(t) = S^{-1}P(S_{\theta(t)}\mathbf{p})^{-1}\mathbf{v}(t)$, we obtain, using (30) and (27)

$$\begin{aligned}
d'(S_t\mathbf{x}) &\leq 2\,p_2\|\mathbf{w}(t)\|\,\|S^{-1}\|\|\mathbf{v}(t)\|\,[c_5(c_3+\|B\|)\|\mathbf{v}(t)\|+c_4]\\
&\quad +\mathbf{w}(t)^*\left[S^{-1}BS+S^*B^*(S^{-1})^*\right]\mathbf{w}(t)\\
&\leq 2p_1p_2\|S\|\,\|S^{-1}\|\,[c_5(c_3+\|B\|)\|\mathbf{v}(t)\|+c_4]\,\|\mathbf{w}(t)\|^2\\
&\quad +\mathbf{w}(t)^*\left[A+A^*\right]\mathbf{w}(t)\\
&\leq 2\epsilon\,\|\mathbf{w}(t)\|^2+\mathbf{w}(t)^*\left[A+A^*\right]\mathbf{w}(t)
\end{aligned}$$

by (24) and (22). Noting that

$$w_1(t) = \mathbf{e}_1^*\mathbf{w}(t) = \mathbf{f}(S_{\theta(t)}\mathbf{p})^*(P^{-1}(S_{\theta(t)}\mathbf{p}))^*(S^{-1})^*S^{-1}P^{-1}(S_{\theta(t)}\mathbf{p})\mathbf{v}(t) = 0$$

we have with Corollary 1

$$\mathbf{w}(t)^*\left[A+A^*\right]\mathbf{w}(t) \leq 2(-\nu+\epsilon)\|\mathbf{w}(t)\|^2.$$

Altogether, we have

$$\begin{aligned}
d'(S_t\mathbf{x}) &\leq [2\epsilon-2\nu+2\epsilon]\,\|\mathbf{w}(t)\|^2\\
&= 2(-\nu+2\epsilon)d(S_t\mathbf{x}),
\end{aligned}$$

which shows $d(S_t\mathbf{x}) \leq e^{2(-\nu+2\epsilon)t}d(\mathbf{x})$ for all $\mathbf{x}\in U$ and $t \geq 0$. ☐

Let us summarize the results obtained so far in the following corollary.

Corollary 2. *Let Ω be an exponentially stable periodic orbit of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f} \in C^\sigma(\mathbb{R}^n,\mathbb{R}^n)$ and $\sigma \geq 2$, such that $-\nu < 0$ is the maximal real part of all non-trivial Floquet exponents.*

For $\epsilon_0 \in (0,\min(\nu,1)) > 0$ there is a compact, positively invariant neighborhood U of Ω with $\Omega \subset U^\circ$ and $U \subset A(\Omega)$, and a map $\pi \in C^{\sigma-1}(U,\Omega)$ with $\pi(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x}\in\Omega$.

Furthermore, for a fixed $\mathbf{x}\in U$, there is a bijective $C^{\sigma-1}$ map $\theta_{\mathbf{x}}\colon [0,\infty)\to[0,\infty)$ with inverse $t_{\mathbf{x}} = \theta_{\mathbf{x}}^{-1} \in C^{\sigma-1}([0,\infty),[0,\infty))$ such that $\theta_{\mathbf{x}}(0) = 0$ and

$$\pi(S_t\mathbf{x}) = S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x})$$

for all $t \in [0,\infty)$. We have $\dot{\theta}_{\mathbf{x}}(t) \in [1-\epsilon_0,1+\epsilon_0]$ for all $t \geq 0$ and $\dot{t}_{\mathbf{x}}(\theta) \in [1-\epsilon_0,1+\epsilon_0]$ for all $\theta \geq 0$.

Finally, there is a constant $C > 0$ such that

$$\begin{aligned}
|\dot{t}_{\mathbf{x}}(\theta)-1| &\leq Ce^{(-\nu+\epsilon_0)\theta} &(31)\\
\|S_{t_{\mathbf{x}}(\theta)}\mathbf{x}-S_\theta\pi(\mathbf{x})\| &\leq Ce^{(-\nu+\epsilon_0)\theta}\|\mathbf{x}-\pi(\mathbf{x})\| &(32)
\end{aligned}$$

for all $\theta \geq 0$ and all $\mathbf{x}\in U$.

Proof. Setting $\epsilon := \frac{\epsilon_0}{2(1+\nu)} \leq \min\left(\frac{\epsilon_0}{2},\frac{1}{2}\right) \leq \min\left(\frac{\nu}{2},1\right)$, all results follow directly from Lemma 3.5 by using the inverse $t(\theta)$ of $\theta(t)$. Indeed, we have

$$\begin{aligned}
|\dot{t}_{\mathbf{x}}(\theta)-1| &= \left|\frac{1-\dot{\theta}_{\mathbf{x}}(t(\theta))}{\dot{\theta}_{\mathbf{x}}(t(\theta))}\right|\\
&\leq \frac{\epsilon}{1-\epsilon}\\
&\leq 2\epsilon \leq \epsilon_0\,.
\end{aligned}$$

Furthermore, we have by (30) and noting that $m_1\|S_{t_{\mathbf{x}}(\theta)}\mathbf{x}-S_\theta\pi(\mathbf{x})\|^2 \leq d(S_{t_{\mathbf{x}}(\theta)}\mathbf{x}) \leq m_2\|S_{t_{\mathbf{x}}(\theta)}\mathbf{x}-S_\theta\pi(\mathbf{x})\|^2$

$$\begin{aligned}
|\dot t_{\mathbf{x}}(\theta)-1| &\leq \left|\frac{1-\dot\theta_{\mathbf{x}}(t(\theta))}{1/2}\right|\\
&\leq 2c_5\|\mathbf{v}(t(\theta))\|\\
&\leq \frac{2c_5}{\sqrt{m_1}}\sqrt{d(S_{t(\theta)}\mathbf{x})}\\
&\leq Ce^{(-\nu+2\epsilon)t(\theta)}\sqrt{d(\mathbf{x})}\\
&\leq Ce^{(-\nu+2\epsilon)(1-2\epsilon)\theta}\\
&\leq Ce^{(-\nu+2\epsilon(1+\nu)-4\epsilon^2)\theta}\\
&\leq Ce^{(-\nu+\epsilon_0)\theta},
\end{aligned}$$

using $t(\theta)=\int_0^\theta \dot t(\tau)\,d\tau \geq \theta(1-2\epsilon)$ and that $d(\mathbf{x})$ is bounded in U. Similarly, we can prove (32) from Lemma 3.5. □

V. Definition of M_1 and M in $A(\Omega)$

For all $\mathbf{x}\in U$ we have defined the distance

$$d(\mathbf{x})=(\mathbf{x}-\pi(\mathbf{x}))^T M_0(\pi(\mathbf{x}))(\mathbf{x}-\pi(\mathbf{x}))$$

in Lemma 3.5 which is $C^{\sigma-1}$. Let $\iota>0$ be so small that the set $\Omega_{2\iota}:=\{\mathbf{x}\in U\colon d(\mathbf{x})\leq 2\iota\}$ satisfies $\Omega_{2\iota}\subset U^\circ$. Define the C^∞ functions $h_1\colon\Omega_\iota\to[0,1]$, $h_2\colon\Omega_{2\iota}\to[0,1]$ such that $h_1(\mathbf{x})=1$ for all $d(\mathbf{x})\leq\frac{\iota}{3}$ and $h_1(\mathbf{x})=0$ for all $d(\mathbf{x})\geq\frac{2}{3}\iota$, and $h_2(\mathbf{x})=1$ for all $d(\mathbf{x})\leq\frac{4}{3}\iota$ and $h_2(\mathbf{x})=0$ for all $d(\mathbf{x})\geq\frac{5}{3}\iota$. Set

$$M_1(\mathbf{x}):=\begin{cases} I & \text{if } \mathbf{x}\notin\Omega_{2\iota},\\ (1-h_2(\mathbf{x}))I+h_2(\mathbf{x})M_0(\pi(\mathbf{x})) & \text{if } \mathbf{x}\in\Omega_{2\iota}.\end{cases}$$

It is clear that $M_1(\mathbf{x})$ is positive definite for all $\mathbf{x}\in\mathbb{R}^n$, M_1 is $C^{\sigma-1}$ and $M_1(\pi(\mathbf{x}))=M_0(\pi(\mathbf{x}))$ for all $\mathbf{x}\in\Omega_{\frac{4}{3}\iota}$.

We will define the Riemannian metric M through M_1 and a scalar-valued function $V\colon A(\Omega)\to\mathbb{R}$, which will be defined later. Let us denote $\mu:=\nu-\epsilon>0$. The function V will be continuous and its orbital derivative V' exists and is continuous. It satisfies

$$\begin{aligned}
V'(\mathbf{x}) &= -L_{M_1}(\mathbf{x})+r(\mathbf{x}),\ \text{where} && (33)\\
r(\mathbf{x}) &= \begin{cases} -\mu & \text{if } \mathbf{x}\notin\Omega_\iota,\\ -\mu(1-h_1(\mathbf{x}))+h_1(\mathbf{x})L_{M_1}(\pi(\mathbf{x})) & \text{if } \mathbf{x}\in\Omega_\iota.\end{cases} && (34)
\end{aligned}$$

Note that

$$r(\mathbf{x})\leq-\mu$$

for all $\mathbf{x}\in\mathbb{R}^n$. Indeed, for $\mathbf{x}\in\Omega_\iota$ we have $L_{M_1}(\pi(\mathbf{x}))=L_{M_0}(\pi(\mathbf{x}))\leq-\mu$ as $\pi(\mathbf{x})\in\Omega$, see (15), and thus

$$r(\mathbf{x}) = -\mu+\underbrace{h_1(\mathbf{x})}_{\geq 0}\underbrace{(\mu+L_{M_1}(\pi(\mathbf{x}))}_{\leq 0}\ \leq\ -\mu\,.$$

Then we define

$$M(\mathbf{x})=e^{2V(\mathbf{x})}M_1(\mathbf{x}).$$

We obtain by Lemma 2.1

$$L_M(\mathbf{x}) = L_{M_1}(\mathbf{x}) + V'(\mathbf{x}) = L_{M_1}(\mathbf{x}) - L_{M_1}(\mathbf{x}) + r(\mathbf{x}) \leq -\mu.$$

This shows the theorem. In the last steps we will define the function V and prove the properties stated above.

VI. Definition of V_{loc}

We define $V_{loc}(\mathbf{x})$ for $\mathbf{x} \in \Omega_\iota$. Note that Ω_ι is positively invariant by Lemma 3.5, so $S_t\mathbf{x} \in \Omega_\iota$ for all $t \geq 0$. We define

$$V_{loc}(\mathbf{x}) \quad = \quad \int_0^\infty [L_{M_1}(S_t\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))]\, dt. \tag{35}$$

We have $V_{loc}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$. We will show that V_{loc} is well-defined, continuous, its orbital derivative V' exists and is continuous for all $\mathbf{x} \in \Omega_\iota$ and that (33) holds for all $\mathbf{x} \in \overline{\Omega_{\iota/3}}$.

For $\mathbf{x} \in U$, define

$$g_T(\tau, \mathbf{x}) \quad = \quad \int_\tau^{T+\tau} [L_{M_1}(S_t\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))]\, dt.$$

By Lemma 3.5 there is a constant $C > 0$ such that, defining $\mathbf{p} := \pi(\mathbf{x}) \in \Omega$,

$$\|S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\mathbf{p}\| \quad \leq \quad Ce^{-\mu_0 t} \tag{36}$$

for all $t \geq 0$ and all $\mathbf{x} \in U$ with $\mu_0 := \nu - 2\epsilon > 0$; note that $S_{\theta_{\mathbf{x}}(t)}\mathbf{p} = \pi(S_t\mathbf{x})$ by (19).

Now, we use Lemma A.1 and $\sigma \geq 3$, showing that L_{M_1} is Lipschitz-continuous on the compact set U; note that $\sigma - 1 \geq 2$. Hence,

$$\begin{aligned} \left|L_{M_1}(S_t\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))\right| \quad &\leq \quad C_1 \left\|S_t\mathbf{x} - S_{\theta_{\mathbf{x}}(t)}\mathbf{p}\right\| \\ &\leq \quad C_2 e^{-\mu_0 t} \end{aligned}$$

by (36), which is integrable over $[0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, the function $g_T(\tau, \mathbf{x})$ converges point-wise for $T \to \infty$ for all $\tau \geq 0$ and $\mathbf{x} \in U$.

Choose $\theta_0 > 0$ so small that $S_{-\theta_0}\Omega_\iota \subset U$. We have that

$$\begin{aligned} &\frac{\partial}{\partial \tau} g_T(\tau, \mathbf{x}) \\ &\quad = \quad [L_{M_1}(S_{T+\tau}\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(T+\tau)}\pi(\mathbf{x}))] - \big(L_{M_1}(S_\tau\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(\tau)}\mathbf{p})\big) \\ &\quad = \quad [L_{M_1}(S_T(S_\tau\mathbf{x})) - L_{M_1}(S_{\theta_{S_\tau\mathbf{x}}(T)}\pi(S_\tau\mathbf{x}))] - \big(L_{M_1}(S_\tau\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(\tau)}\mathbf{p})\big) \end{aligned}$$

by Lemma 3.4. For $\mathbf{x} \in \Omega_\iota$, the right-hand side converges uniformly in $\tau \in (-\theta_0, \theta_0)$ as $T \to \infty$ to $-\big(L_{M_1}(S_\tau\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(\tau)}\mathbf{p})\big)$ by the same estimate as above. Hence, we can commute $\frac{d}{d\tau}$ and $\lim_{T\to\infty}$. Altogether, we thus have for all $\mathbf{x} \in \Omega_\iota$, using

Lemma 3.4

$$\begin{aligned}
V'_{loc}(\mathbf{x}) &= \frac{d}{d\tau}V_{loc}(S_\tau\mathbf{x})\Big|_{\tau=0}\\
&= \frac{d}{d\tau}\int_0^\infty [L_{M_1}(S_{t+\tau}\mathbf{x}) - L_{M_1}(S_{\theta_{S_\tau\mathbf{x}}(t)}\pi(S_\tau\mathbf{x}))]\,dt\Big|_{\tau=0}\\
&= \frac{d}{d\tau}\lim_{T\to\infty}\int_0^T [L_{M_1}(S_{t+\tau}\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(t+\tau)}\pi(\mathbf{x}))]\,dt\Big|_{\tau=0}\\
&= \frac{d}{d\tau}\lim_{T\to\infty}\int_\tau^{T+\tau} [L_{M_1}(S_t\mathbf{x}) - L_{M_1}(S_{\theta_{\mathbf{x}}(t)}\pi(\mathbf{x}))]\,dt\Big|_{\tau=0}\\
&= \frac{d}{d\tau}\lim_{T\to\infty} g_T(\tau,\mathbf{x})\Big|_{\tau=0}\\
&= \lim_{T\to\infty}\frac{d}{d\tau} g_T(\tau,\mathbf{x})\Big|_{\tau=0}\\
&= -L_{M_1}(\mathbf{x}) + L_{M_1}(\mathbf{p})
\end{aligned}$$

and in particular, that V'_{loc} exists and is continuous. Note that $V'_{loc}(\mathbf{x}) = -L_{M_1}(\mathbf{x}) + r(\mathbf{x})$ for all $\mathbf{x}\in\overline{\Omega_{\iota/3}}$.

VII. Definition of V_{glob} in $A(\Omega)$

For the global part, note that V_{loc} is defined and smooth in Ω_ι and we have $V'_{loc}(\mathbf{x}) = -L_{M_1}(\mathbf{x})+r(\mathbf{x})$ for all $\mathbf{x}\in\overline{\Omega_{\iota/3}}$. The global function $V_{glob}\colon A(\Omega)\setminus\Omega\to\mathbb{R}$ is defined as the solution of the non-characteristic Cauchy problem

$$\begin{aligned}
\nabla V_{glob}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}) &= -L_{M_1}(\mathbf{x}) + r(\mathbf{x}) \text{ for } \mathbf{x}\in A(\Omega)\setminus\Omega\\
V_{glob}(\mathbf{x}) &= V_{loc}(\mathbf{x}) \text{ for } \mathbf{x}\in\Gamma,
\end{aligned}\tag{37}$$

where $\Gamma = \{\mathbf{x}\in U \mid d(\mathbf{x}) = \iota/3\}$. It is clear that V_{glob} is continuous, and V'_{glob} exists and is continuous on $A(\Omega)\setminus\Omega$.

In particular, we can construct the solution by first defining the function $\tau\in C^\sigma(A(\Omega)\setminus\Omega,\mathbb{R})$ implicitly by

$$d(S_\tau\mathbf{x}) = \iota/3.$$

Since $\mathbf{x}\in A(\Omega)\setminus\Omega$, there exists a τ satisfying the equation, and since $d'(\mathbf{x})<0$ for all $\mathbf{x}\in\Gamma$, $\tau(\mathbf{x})$ is unique. The function τ is $C^{\sigma-1}$, since d and S_τ are. We have $\tau'(\mathbf{x}) = -1$. Then the function

$$V_{glob}(\mathbf{x}) = \int_0^{\tau(\mathbf{x})} q(S_t\mathbf{x})\,dt + V_{loc}(S_{\tau(\mathbf{x})}(\mathbf{x}))$$

with $q(\mathbf{x}) := L_{M_1}(\mathbf{x}) - r(\mathbf{x})$ is continuous, its orbital derivative V'_{glob} exists and is continuous, and V_{glob} satisfies (37), noting that $S_{\tau(\mathbf{x})}(\mathbf{x}) = S_{\tau(S_\theta\mathbf{x})}(S_\theta\mathbf{x})$ for all

$\theta \geq 0$. Indeed, for $\mathbf{x} \in \Gamma$ we have $V_{glob}(\mathbf{x}) = V_{loc}(\mathbf{x})$ and we have

$$\begin{aligned} V'_{glob}(\mathbf{x}) &= \frac{d}{d\theta}\left(\int_0^{\tau(S_\theta \mathbf{x})} q(S_{t+\theta}\mathbf{x})\,dt + V_{loc}(S_{\tau(S_\theta \mathbf{x})}(S_\theta \mathbf{x}))\right)\Big|_{\theta=0} \\ &= \frac{d}{d\theta}\left(\int_\theta^{\tau(S_\theta \mathbf{x})+\theta} q(S_t\mathbf{x})\,dt + V_{loc}(S_{\tau(\mathbf{x})}(\mathbf{x}))\right)\Big|_{\theta=0} \\ &= \left(q(S_{\tau(S_\theta \mathbf{x})+\theta}\mathbf{x})(\tau'(\mathbf{x})+1) - q(S_\theta \mathbf{x})\right)\Big|_{\theta=0} \\ &= -q(\mathbf{x}) \end{aligned}$$

since $\tau'(\mathbf{x}) = -1$.

Note that we have $V_{glob}(\mathbf{x}) = V_{loc}(\mathbf{x})$ for $\mathbf{x} \in \overline{\Omega_{\iota/3}} \setminus \Omega$, and hence V_{glob} can be extended to a continuous function V on $A(\Omega)$ by setting $V_{glob}(\mathbf{x}) := V_{loc}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$. Then also its orbital derivative V'_{glob} exists and is continuous and V_{glob} satisfies (33). This proves the theorem. □

Conclusions. In this paper we have studied contraction metrics, which are Riemannian metrics on the phase space $\mathbb{R}^n$. Moreover, the distance, defined by the induced norm, of adjacent solution trajectories is decreasing over time. Here, only adjacent solutions in direction perpendicular to the flow with respect to the induced scalar product are considered. The existence of such a contraction metric in a compact, connected and positively invariant set, which contains no equilibrium, implies the existence of a unique periodic orbit as well as its exponential stability. Moreover, it provides an upper bound on the non-trivial Floquet exponents and determines a subset of the basin of attraction of the periodic orbit.

This paper has considered the converse question, namely the existence of such a contraction metric. We have proved the existence of a contraction metric for an exponentially stable periodic orbit in its basin of attraction, and the upper bound on the function L_M is arbitrarily close to the true exponential rate of attraction. The construction is achieved by first defining the contraction metric on the periodic orbit, then in a neighborhood and finally in the whole basin of attraction as the solution of a non-characteristic Cauchy problem.

Appendix A. **Local Lipschitz-continuity of L_M.** In the appendix we prove that the function L_M is locally Lipschitz continuous.

Lemma A.1. *Let $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $M \in C^2(\mathbb{R}^n, \mathbb{S}^n)$ such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$.*

Then L_M is locally Lipschitz continuous on $D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \neq \mathbf{0}\}$.

Proof. For $\mathbf{y} \in D$ we define a projection $P_{\mathbf{y}} : \mathbb{R}^n \to \mathbb{R}^n$ onto the $(n-1)$-dimensional space of vectors $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{w} = 0$ by

$$P_{\mathbf{y}}\mathbf{v} = \mathbf{v} - \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\mathbf{f}(\mathbf{y})$$

for all $\mathbf{y} \in D$ and all $\mathbf{v} \in \mathbb{R}^n$. Note that indeed

$$\begin{aligned} \mathbf{f}(\mathbf{y})^T M(\mathbf{y}) P_{\mathbf{y}}\mathbf{v} &= \mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v} - \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y}) \\ &= \mathbf{0}. \end{aligned}$$

Fix $\mathbf{x} \in D$ and choose a basis $\mathbf{v}_1 = \mathbf{f}(\mathbf{x}), \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ such that $\mathbf{v}_i^T M(\mathbf{x})\mathbf{v}_j = 0$ for $i \neq j$. Choose $\epsilon > 0$ such that

$$\mathbf{f}(\mathbf{y})^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) \neq 0 \tag{38}$$

holds for all $\mathbf{y} \in B_\epsilon(\mathbf{x})$; note that for $\mathbf{y} = \mathbf{x}$ we have $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{x}) \neq 0$.

For $\mathbf{y} \in B_\epsilon(\mathbf{x})$ we define $\mathbf{w}_1 = \mathbf{f}(\mathbf{y})$ and $\mathbf{w}_i = P_{\mathbf{y}}\mathbf{v}_i$ for $i = 2, \ldots, n$. We show that $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ is a basis of $\mathbb{R}^n$.

Let us first show that $\mathbf{w}_i \neq \mathbf{0}$ for $i = 2, \ldots, n$. Assuming the opposite, we have

$$\begin{aligned} \mathbf{v}_i &= \frac{\mathbf{f}(\mathbf{y})M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\mathbf{f}(\mathbf{y}) \qquad (39)\\ 0 &= \frac{\mathbf{f}(\mathbf{y})M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})} \end{aligned}$$

multiplying by $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})$ from the left as $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{y}) \neq 0$ by (38). This, however, implies by (39) that $\mathbf{v}_i = \mathbf{0}$ which is a contradiction. $\mathbf{w}_1 \neq \mathbf{0}$ follows directly from (38).

We express $\mathbf{f}(\mathbf{y}) = \sum_{j=1}^n \beta_j \mathbf{v}_j$ and note that multiplying this equation by $\mathbf{f}(\mathbf{x})^T M(\mathbf{x})$ from the left gives

$$0 \neq \mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{y}) = \beta_1 \mathbf{f}(\mathbf{x})^T M(\mathbf{x})\mathbf{f}(\mathbf{x})$$

by (38), i.e. in particular $\beta_1 \neq 0$.

To show that the $\mathbf{w}_i$ form a basis, we assume $\sum_{i=1}^n \alpha_i \mathbf{w}_i = \mathbf{0}$. Multiplying this equation by $\mathbf{f}(\mathbf{y})^T M(\mathbf{y})$ from the left gives $\alpha_1 \mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y}) = 0$ by the projection property, hence $\alpha_1 = 0$.

Hence,

$$\begin{aligned} \mathbf{0} &= \sum_{i=2}^n \alpha_i \left[\mathbf{v}_i - \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\mathbf{f}(\mathbf{y})\right] \\ &= \sum_{i=2}^n \alpha_i \mathbf{v}_i - \sum_{i=2}^n \sum_{j=1}^n \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\beta_j \mathbf{v}_j. \end{aligned}$$

Using that $\mathbf{v}_j$ is a basis, we can conclude that the coefficient in front of $\mathbf{v}_1$ is zero, namely

$$\sum_{i=2}^n \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})}\beta_1 = 0.$$

Since $\beta_1 \neq 0$, we have $\sum_{i=2}^n \frac{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{v}_i}{\mathbf{f}(\mathbf{y})^T M(\mathbf{y})\mathbf{f}(\mathbf{y})} = 0$. Plugging this back in, we obtain $\sum_{i=2}^n \alpha_i \mathbf{v}_i = \mathbf{0}$, which shows $\alpha_2 = \ldots = \alpha_n = 0$ as the $\mathbf{v}_i$ are linearly independent.

Now define the matrix-valued function $Q\colon B_\epsilon(\mathbf{x}) \to \mathbb{R}^{n\times n}$ by the columns

$$Q(\mathbf{y}) = (\mathbf{w}_1(\mathbf{y}), \ldots, \mathbf{w}_n(\mathbf{y})).$$

Note that $Q \in C^2(B_\epsilon(\mathbf{x}), \mathbb{R}^{n\times n})$ due to the smoothness of $\mathbf{f}$ and M, and Q is invertible for every $\mathbf{y}$. We have $\mathbf{w}^T M(\mathbf{y})\mathbf{f}(\mathbf{y}) = 0$ if and only if $\mathbf{w} \in \operatorname{span}(\mathbf{w}_2(\mathbf{y}), \ldots, \mathbf{w}_n(\mathbf{y})$, which in turn is equivalent to $\mathbf{u} \in \operatorname{span}(\mathbf{e}_2, \ldots, \mathbf{e}_n) =: E_{n-1}$, where $\mathbf{u} = Q(\mathbf{y})^{-1}\mathbf{w}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis in $\mathbb{R}^n$.

Now we write

$$
\begin{aligned}
&L_M(\mathbf{y})\\
= & \max_{\mathbf{w}^T M(\mathbf{y})\mathbf{w}=1,\mathbf{w}^T M(\mathbf{y})\mathbf{f}(\mathbf{y})=0} \frac{1}{2}\mathbf{w}^T\left[M(\mathbf{y})D\mathbf{f}(\mathbf{y})+D\mathbf{f}(\mathbf{y})^T M(\mathbf{y})+M'(\mathbf{y})\right]\mathbf{w}\\
= & \max_{\mathbf{u}^T Q(\mathbf{y})^T M(\mathbf{y})Q(\mathbf{y})\mathbf{u}=1,\mathbf{u}\in E_{n-1}} \frac{1}{2}\mathbf{u}^T Q(\mathbf{y})^T\\
& \qquad \left[M(\mathbf{y})D\mathbf{f}(\mathbf{y})+D\mathbf{f}(\mathbf{y})^T M(\mathbf{y})+M'(\mathbf{y})\right]Q(\mathbf{y})\mathbf{u}.
\end{aligned}
$$

Denoting by $[A]_{n-1}\in\mathbb{S}^{n-1}$ the lower-right square $(n-1)$ matrix of $A\in\mathbb{S}^n$ and with $\mathbf{u}=\begin{pmatrix}0\\ \widetilde{\mathbf{u}}\end{pmatrix}$, where $\widetilde{\mathbf{u}}\in\mathbb{R}^{n-1}$, we get

$$
\begin{aligned}
L_M(\mathbf{y}) = & \max_{\widetilde{\mathbf{u}}^T[Q(\mathbf{y})^T M(\mathbf{y})Q(\mathbf{y})]_{n-1}\widetilde{\mathbf{u}}=1,\widetilde{\mathbf{u}}\in\mathbb{R}^{n-1}} \frac{1}{2}\widetilde{\mathbf{u}}^T\Big[Q(\mathbf{y})^T\big[M(\mathbf{y})D\mathbf{f}(\mathbf{y})\\
& \qquad +D\mathbf{f}(\mathbf{y})^T M(\mathbf{y})+M'(\mathbf{y})\big]Q(\mathbf{y})\Big]_{n-1}\widetilde{\mathbf{u}}.
\end{aligned}
$$

Now denote by $\mathrm{Chol}(A)$ the unique Cholesky decomposition of the symmetric, positive definite matrix $A\in\mathbb{S}^{n-1}$, such that $\mathrm{Chol}(A)$ is an invertible, upper triangular matrix with $\mathrm{Chol}(A)^T\,\mathrm{Chol}(A)=A$. Denoting $C(\mathbf{y}):=\mathrm{Chol}([Q(\mathbf{y})^T M(\mathbf{y})Q(\mathbf{y})]_{n-1})\in\mathbb{R}^{(n-1)\times(n-1)}$ and $\widetilde{\mathbf{v}}=C(\mathbf{y})\widetilde{\mathbf{u}}\in\mathbb{R}^{n-1}$ we have

$$
\begin{aligned}
L_M(\mathbf{y}) &= \max_{\|\widetilde{\mathbf{v}}\|=1,\widetilde{\mathbf{v}}\in\mathbb{R}^{n-1}} \frac{1}{2}\widetilde{\mathbf{v}}^T(C^{-1}(\mathbf{y}))^T\\
&\qquad \left[Q(\mathbf{y})^T\left[M(\mathbf{y})D\mathbf{f}(\mathbf{y})+D\mathbf{f}(\mathbf{y})^T M(\mathbf{y})+M'(\mathbf{y})\right]Q(\mathbf{y})\right]_{n-1}C^{-1}(\mathbf{y})\widetilde{\mathbf{v}}\\
&= \max_{\|\widetilde{\mathbf{v}}\|=1,\widetilde{\mathbf{v}}\in\mathbb{R}^{n-1}} \widetilde{\mathbf{v}}^T H(\mathbf{y})\widetilde{\mathbf{v}}\\
&= \lambda_{max}(H(\mathbf{y})),
\end{aligned}
$$

where $\lambda_{max}(S)$ denotes the maximal eigenvalue of the symmetric matric $S\in\mathbb{S}^{n-1}$ and $H(\mathbf{y})\in\mathbb{S}^{n-1}$ is defined by

$$
\begin{aligned}
H(\mathbf{y}) &= \frac{1}{2}(C^{-1}(\mathbf{y}))^T\left[Q(\mathbf{y})^T\left[M(\mathbf{y})D\mathbf{f}(\mathbf{y})+D\mathbf{f}(\mathbf{y})^T M(\mathbf{y})+M'(\mathbf{y})\right]Q(\mathbf{y})\right]_{n-1}\\
&\quad C^{-1}(\mathbf{y}).
\end{aligned}
$$

The function $\mathbf{y}\to H(\mathbf{y})$ is continuously differentiable as the Cholesky decomposition, the inverse, the operation $[\cdot]_{n-1}$, Q, M, $D\mathbf{f}$ and M' are continuously differentiable by the assumptions. Hence, the function $H(\mathbf{y})$ is locally Lipschitz-continuous. The function λ_{max} is globally Lipschitz-continuous, hence, L_M is locally Lipschitz-continuous. □

REFERENCES

[1] V. A. Boichenko and G. A. Leonov. Lyapunov orbital exponents of autonomous systems. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 3:7–10, 123, 1988.

[2] G. Borg. *A condition for the existence of orbitally stable solutions of dynamical systems*, volume 153. Elander, 1960.

[3] C. Chicone. *Ordinary Differential Equations with Applications.* New York: Springer-Verlag, 2006.

[4] F. Forni and R. Sepulchre. A differential Lyapunov framework for Contraction Analysis. *IEEE Trans. Automat. Control*, 59(3):614–628, 2014.

[5] P. Giesl. On a matrix-valued PDE characterizing a contraction metric for a periodic orbit. submitted.

[6] P. Giesl. Necessary conditions for a limit cycle and its basin of attraction. *Nonlinear Anal.*, 56:643–677, 2004.

[7] P. Giesl. Converse theorems on contraction metrics for an equilibrium. *J. Math. Anal. Appl.*, 424:1380–1403, 2015.

[8] P. Hartman. *Ordinary Differential Equations*. Wiley, New York, 1964.

[9] P. Hartman and C. Olech. On global asymptotic stability of solutions of differential equations. *Trans. Amer. Math. Soc.*, 104:154–178, 1962.

[10] A. Yu. Kravchuk, G. A. Leonov, and D. V. Ponomarenko. Criteria for strong orbital stability of trajectories of dynamical systems. I. *Differentsialnye Uravneniya*, 28(9):1507–1520, 1652, 1992.

[11] G. A. Leonov, I. M. Burkin, and A. I. Shepelyavyi. *Frequency Methods in Oscillation Theory*. Ser. Math. and its Appl.: Vol. 357, Kluwer, 1996.

[12] W. Lohmiller and J.-J. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34:683–696, 1998.

[13] B. Stenström. Dynamical systems with a certain local contraction property. *Math. Scand.*, 11:151–155, 1962.

[14] S. Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*. New York: Springer-Verlag, 1994.

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